TOPOLOGICAL PROPERTIES OF THE EFFICIENT POINT SET

BEZALEL PELEG

Abstract. Let \( Y \) be a closed and convex subset of a Euclidean space. We prove that the set of efficient points of \( Y \), \( M(Y) \), is contractible. Furthermore, if \( M(Y) \) is closed (compact) then it is a retract of a convex closed (compact) set. Our proof relies on the Arrow-Barankin-Blackwell Theorem. A new proof is supplied for that theorem.

1. Introduction. The study of efficient points of convex sets is expounded by many writers (see, e.g., [3], [2, pp. 306–310], [6, §12.3]). In particular, topological properties of the efficient point set are investigated in [3, pp. 73–78]. This paper is a further contribution on this topic: In §4 we prove that the set of efficient points \( M(Y) \) of a closed and convex subset \( Y \) of a Euclidean space is contractible. Furthermore, if \( M(Y) \) is closed (compact) then it is a retract of a convex closed (compact) set.

Our proofs make use of the Arrow-Barankin-Blackwell Theorem [1, Theorem 1]. This theorem is generalized in infinite-dimensional spaces in [9], [7], [8], [5], and [4]. In §3 we offer a new proof of the Arrow-Barankin-Blackwell Theorem. Our proof is advantageous over the original one in two respects: It is a “constructive” proof, unlike that of Arrow, Barankin, and Blackwell. Furthermore, it is easier to generalize to infinite-dimensional spaces (see [7]).

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2. Preliminaries. Let \( E^n \) be the \( n \)-dimensional Euclidean space. If \( x, y \in E^n \), then we write \( x \succeq y \) if \( x_i \geq y_i \) for \( i = 1, \cdots, n \). \( x \succ y \) if \( x \succeq y \) and \( x \neq y \). \( x \succcurlyeq y \) if \( x_i > y_i \) for \( i = 1, \cdots, n \). We denote by \( E_+^n \) the nonnegative cone of \( E^n \), i.e., \( E_+^n = \{ x \mid x \in E^n \text{ and } x \succeq 0 \} \). The scalar product of two members \( x \) and \( y \) of \( E^n \) is denoted by \( x \cdot y = \sum_{i=1}^{n} x_i y_i \). The norm of a member \( x \) of \( E^n \) is denoted by \( \| x \| = (x \cdot x)^{1/2} \). \( u^{(i)}, i = 1, \cdots, n \), will denote the \( i \)th unit vector of \( E^n \).

3. An alternative proof of the Arrow-Barankin-Blackwell Theorem. Let \( Y \) be subset of \( E^n \). A point \( e \in Y \) is an efficient point of \( Y \) if there exists no
$y \in Y$ such that $y > e$. $r \in Y$ is a regular efficient point of $Y$ if there exists a vector $p \in E^n$, $p > 0$, such that $p \cdot r \geq p \cdot y$ for all $y \in Y$. Clearly, a regular efficient point of $Y$ is an efficient point of $Y$. Let

$$M(Y) = \{e \mid e \text{ is an efficient point of } Y\}.$$ 

**Theorem 3.1 (Arrow, Barankin, and Blackwell [1]).** Let $Y$ be a closed and convex subset of $E^n$. The regular efficient points of $Y$ are dense in $M(Y)$.

**Proof.** Let $e \in M(Y)$ and let $Y^* = \{y \mid y \in Y \text{ and } \|y - e\| \leq 1\}$. If $r^*$ is a regular efficient point of $Y^*$ and $\|r^* - e\| < 1$ then $r^*$ is a regular efficient point of $Y$. To see this let $p > 0$ satisfy $p \cdot r^* \geq p \cdot y^*$ for all $y^* \in Y^*$. Let $y \in Y$. For $t > 0$ sufficiently small, $ty + (1-t)r^* \in Y^*$. Hence, $p \cdot r^* \geq t \cdot p \cdot y + (1-t)p \cdot r^*$. Thus, $p \cdot r^* \geq p \cdot y$. Thus, it is sufficient to prove that $e$ is the limit of a sequence of regular efficient points of $Y^*$. But $Y^*$ is compact. Hence, we may assume that $Y$ is compact. We may assume further that $Y \subseteq E^n$. Let $C = \max\{\|y\| \mid y \in Y\}$. For each $k$, $k = 1, 2, \ldots$, let

$$Y^{(k)} = \{y \mid y \in Y \text{ and } y_i \geq e_i - 1/k, \ i = 1, \ldots, n\},$$

$$v_k(x) = \min(x_i - e_i + 1/k, 1 \leq i \leq n), \quad x \in E^n,$$

$$w_k(x) = \sum_{i=1}^{n} x_i/n(k + 1)C, \quad x \in E^n,$$

and

$$u_k(x) = v_k(x) + w_k(x).$$

Let $r^{(k)}$ be a point where $u_k$ attains its maximum in $Y^{(k)}$. $u_k$ is concave; hence, the set

$$Z = \{z \mid z \in E^n \text{ and } u_k(z) > u_k(r^{(k)})\}$$

is convex. $Z \cap Y^{(k)} = \emptyset$. Hence, there exists a $p \in E^n$ such that

$$p \cdot z \geq p \cdot y \quad \text{for all } z \in Z \text{ and } y \in Y^{(k)}.$$ 

By (3.3), (3.4), and (3.5), $u_k$ is increasing, i.e., $x > y$ implies that $u_k(x) > u_k(y)$. Hence, it follows from (3.6) and (3.7) that $p > 0$. Furthermore,

$$p \cdot r^{(k)} \geq p \cdot y \quad \text{for all } y \in Y^{(k)}.$$ 

Since $1/k \leq u_k(e) \leq u_k(r^{(k)})$, it follows from (3.4) that

$$r_i^{(k)} > e_i - 1/k, \quad i = 1, \ldots, n.$$
It follows from (3.8) and (3.9) that
\[(3.10) \quad p \cdot r^{(k)} \geq p \cdot y \quad \text{for all } y \in Y.\]
Thus, \(r^{(k)}\) is a regular efficient point of \(Y\). Since \(r^{(k)} \in Y^{(k)}, k = 1, 2, \ldots\),
and \(e\) is efficient, \(e = \lim_{k \to \infty} r^{(k)}\).

4. A proof that the set of efficient points is contractible. Let \(Y\) be a closed and convex subset of \(E^n\) and let \(M(Y) \neq \emptyset\) (see (3.1)).

**Lemma 4.1.** There exist a vector \(p \gg 0\) and a real number \(v\) such that \(p \cdot y \leq v\) for all \(y \in Y\).

**Proof.** By Theorem 3.1 there exists a regular efficient point of \(Y\).

**Corollary 4.2.** For each \(x \in E^n\) the set \(\{y \mid y \in Y \text{ and } y \geq x\}\) is compact.

**Lemma 4.3.** Let \(Y^* = \{x \mid \text{there exists } y \in Y \text{ such that } y \geq x\}\). Then \(Y^*\) is convex and closed and \(M(Y) = M(Y^*)\).

**Proof.** It is clear that \(Y^*\) is convex and that \(M(Y) = M(Y^*)\). To see that \(Y^*\) is closed let \(x = \lim_{k \to \infty} x^{(k)}, x^{(k)} \in Y^*, k = 1, 2, \ldots\). There exist \(y^{(k)} \in Y, y^{(k)} \geq x^{(k)}, k = 1, 2, \ldots\). By Corollary 4.2 the sequence \((y^{(k)})\) is bounded. Hence, we may assume that there exists a vector \(y\) such that \(y = \lim_{k \to \infty} y^{(k)}\). Clearly, \(y \in Y\) and \(y \geq x\).

By Lemma 4.3 we may assume henceforth that \(Y = Y^*\).

**Corollary 4.4.** There exist points \(a, b \in Y\) such that \(b \gg a\).

For \(y \in Y\) we define
\[(4.1) \quad G(y) = \{x \mid x \in Y \text{ and } x \geq y\}.\]

\(G(y)\) is convex and compact. Also, \(G\) is an upper semicontinuous function of \(y\).

**Lemma 4.5.** Let \(y \in Y\). If there exists a \(z \in Y\) such that \(z \gg y\) then \(G\) is lower semicontinuous at \(y\).

**Proof.** Let \(y = \lim_{k \to \infty} y^{(k)}\) and let \(x \in G(y)\). Let \(1 > t > 0\). \(x(t) = tz + (1 - t)x \gg y\).

Hence, there exists a natural number \(k(t)\) such that \(x(t) = G(y^{(k)}\) for \(k \geq k(t)\). Since \(\lim_{t \to 0} x(t) = x\), the lemma follows.

We recall that a topological space is contractible if its identity map is homotopic to a constant.

**Theorem 4.6.** \(M(Y)\) is contractible.

**Proof.** Let \(p\) and \(v\) be as in Lemma 4.1. Let \(w = \min_{1 \leq i \leq n} p_i\). For \(y \in Y\) let \(f(y) = y + (v - p \cdot y) \sum_{i=1}^{n} d(i)^{1/w}\). \(f\) is a continuous function of \(y\).
Furthermore,

\begin{equation}
(4.2) \quad f(y) \geq x \quad \text{for all } x \in G(y) \quad \text{(see (4.1)).}
\end{equation}

For \( y \in Y \) let \( g(y) \in G(y) \) be the point defined by \( \|g(y) - f(y)\| \leq \|x - f(y)\| \)
for all \( x \in G(y) \). \( g(y) \) is well defined. By (4.2), \( g(y) \in M(Y) \). Now let \( a \in Y \)
be a point for which there exists a \( b \in Y \) such that \( b \gg a \) (see Corollary 4.4). For \( e \in M(Y) \)
and \( 0 \leq t \leq 1 \) let \( h(e, t) = g((1 - t)e + ta) \). \( h(e, 0) = e \) and \( h(e, 1) = g(a) \)
for all \( e \in M(Y) \). Furthermore, \( h \) is a continuous function of both \( e \) and \( t \). For let \( t = \lim_{k \to \infty} f^{(k)} \) and \( e = \lim_{k \to \infty} e^{(k)} \). If \( t = 0 \) then

\begin{equation}
\lim_{k \to \infty} (1 - t^{(k)})e^{(k)} + t^{(k)}a = e.
\end{equation}

(4.3) \quad \begin{cases} \ h(e^{(k)}, t^{(k)}) \geq (1 - t^{(k)})e^{(k)} + t^{(k)}a. \end{cases}

By Corollary 4.2 the sequence \((h(e^{(k)}, t^{(k)}))\) is bounded. Hence, it follows from (4.3) and from \( e \in M(Y) \) that \( \lim_{k \to \infty} h(e^{(k)}, t^{(k)}) = e \). If \( t > 0 \) then

\begin{equation}
(1 - t)e + ta \gg (1 - t)e + ta = \lim_{k \to \infty} (1 - t^{(k)})e^{(k)} + t^{(k)}a.
\end{equation}

Hence, by Lemma 4.5, \( G \) is lower semicontinuous at \((1 - t)e + ta\). Therefore, \( g \) is continuous at \((1 - t)e + ta = y\). For assume, on the contrary, that

\begin{equation}
(4.4) \quad y = \lim_{k \to \infty} y^{(k)} \quad \text{and} \quad \lim_{k \to \infty} g(y^{(k)}) = z \neq g(y).
\end{equation}

Then

\begin{equation}
\|g(y) - f(y)\| < \|z - f(y)\|.
\end{equation}

Furthermore, there exist \( x^{(k)} \in G(y^{(k)}), \ k = 1, 2, \ldots \), such that

\begin{equation}
\lim_{k \to \infty} x^{(k)} = g(y).
\end{equation}

It follows from (4.4)–(4.6) that there exists a \( k \) such that

\begin{equation}
\|x^{(k)} - f(y^{(k)})\| < \|g(y^{(k)}) - f(y^{(k)})\|,
\end{equation}

which is impossible. The continuity of \( h \) at \((e, t)\) follows now from the
continuity of \( g \) at \((1 - t)e + ta\).

We recall that a subset \( A \) of a topological space \( T \) is a retract of \( T \) if there exists a continuous function \( r: T \to A \) such that \( r(a) = a \) for all \( a \in A \).

**Theorem 4.7.** If \( M(Y) \) is closed then it is a retract of a closed and convex set.

**Proof.** For \( y \in Y \) let \( d(y, M(Y)) \) be the distance between \( y \) and \( M(Y) \).
Let

\begin{equation}
(4.7) \quad r(y) = d(y, M(Y))/(1 + d(y, M(Y))).
\end{equation}
Then \( t(y) \) is a continuous function of \( y \) and \( y \in M(Y) \) if and only if \( t(y) = 0 \). Using the notation of the proof of Theorem 4.6 we define, for \( y \in Y \),

\[
h(y) = g((1 - t(y))y + t(y)a).
\]

Then \( h(y) \in M(Y) \) and \( h(e) = e \) for \( e \in M(Y) \). Furthermore, it follows from the definition of \( t(y) \) and Lemma 4.5 that \( h \) is continuous. Hence, \( h \) is a retraction of \( Y \) on \( M(Y) \).

**Theorem 4.8.** If \( M(Y) \) is compact then it is a retract of a compact and convex set.

**Proof.** Choose \( a \in E^n \) for which there exists \( e \in M(Y) \) such that \( e \gg a \). Let \( Y_1 \) be the convex hull of \( M(Y) \cup \{a\} \). Then \( Y_1 \) is convex and compact and \( M(Y_1) = M(Y) \). Let \( q \in E^n \) satisfy \( q \geq x \) for all \( x \in Y_1 \). For \( y \in Y_1 \) let \( g(y) \in G(y) = \{x : x \geq y \text{ and } x \in Y_1\} \) be defined by

\[
\|g(y) - q\| \leq \|x - q\| \quad \text{for all } x \in G(y).
\]

Let further \( t(y) \) be defined by (4.7). Define now, for \( y \in Y_1 \),

\[
h(y) = g((1 - t(y))y + t(y)a).
\]

Then \( h \) is a retraction of \( Y_1 \) on \( M(Y) \).

**Remark 4.9.** If \( Y \) is polyhedral or strictly convex then \( M(Y) \) is closed. However, \( Y \) may be compact without \( M(Y) \) being closed.

5. **Concluding remarks.** Let \( Y \) be a closed and convex subset of a Euclidean space. Consider \( Y \) as a technology given in the flow version (see [6, §12]). By Theorem 4.6, \( M(Y) \) is contractible; hence, in particular, it is arcwise connected. Thus, it is possible to move from one efficient process to another via \( M(Y) \) in a continuous manner. This result may be useful for economic planning.

If \( Y \) is compact and polyhedral then, by Theorem 4.8, \( M(Y) \) is a retract of a convex compact set. Therefore, every continuous function \( f : M(Y) \to M(Y) \) has a fixed point. This last fact may prove useful in game theory and mathematical economics investigations.

**References**


Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel