A SPLITTING RING OF GLOBAL DIMENSION TWO

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Abstract. In this paper an example is given of a ring with left
global dimension 2 having the property that the singular sub-
module of any \( \mathcal{R} \)-module \( A \) is a direct summand of \( A \). Although the
example given is quite specific, the methods can be used to construct
a fairly large class of these rings.

In this paper, all rings are assumed to be associative with an identity
element, and all modules will be unitary left modules.

An \( \mathcal{R} \)-module \( A \) is said to split if the singular submodule, \( Z(\mathcal{R}A) \), is a
direct summand of \( A \). \( \mathcal{R} \) is called a splitting ring if every \( \mathcal{R} \)-module splits
(see \([1, 3, \text{and } 7]\)). In \([3]\) Cateforis and Sandomierski have shown that
every commutative splitting ring has left global dimension \( \leq 1 \). M. L.
Teply \([7]\) has shown that if the commutative hypothesis is dropped, then
every splitting ring must have left global dimension \( \leq 2 \). Several splitting
rings of left global dimension 1 were known, but no such rings of left
global dimension 2 have been found; thus the question arises, which is the
best bound, 1 or 2? In this paper it is shown that 2 is the best possible
bound for the left global dimension of a splitting ring.

\( \mathcal{R} \) is said to have the finitely generated splitting property (FGSP)
if every finitely generated \( \mathcal{R} \)-module splits. Cateforis and Sandomierski
\([3]\) have shown that every commutative ring with FGSP must be semi-
ereditary. A trivial consequence of the example constructed in this
paper is that a (noncommutative) ring with FGSP need not be left or
right semihereditary.

For an \( \mathcal{R} \)-module \( A \), let \( \text{soc}(A) \) denote the socle of \( A \). If \( \mathcal{R} \) is a ring of
matrices, we define \( e_{ij} \) to be the matrix with the identity element of the
appropriate coordinate ring in the \( i \)th row and \( j \)th column and zeros
elsewhere.

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Let $S$ be a ring, and let $M$ be an essential maximal left ideal of $S$ which is also a two-sided ideal. Let

$$R = \begin{cases} 
(a & b & c \\
0 & d & e \\
0 & 0 & f 
\end{cases} \quad d \in S; \quad a, b, c, e, f \in S/M,
$$

and let

$$\Lambda = \begin{cases} 
(a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c 
\end{cases} \quad b \in S; \quad a, c \in S/M.
$$

We wish to choose $S$ such that $R$ will become the desired splitting ring of left global dimension two. But first we point out a few basic properties of $R$:

**Lemma 1.** (a) $R$ is a left (right) Noetherian ring if and only if $S$ is a left (right) Noetherian ring.

(b) If $Z(S) = 0$, then $Z(R) = 0$.

(c) If $S$ has no nontrivial idempotent elements, then $\text{l.gl.dim} \ R = 2$.

**Proof.** (a) The "if" part is an immediate consequence of the fact that $SR$ is finitely generated. The "only if" follows from the existence of a (ring) homomorphism of $R$ onto $S$ given by

$$\begin{array}{l}
(a & b & c \\
0 & d & e \\
0 & 0 & f 
\end{array} \rightarrow d.
$$

(b) If $Z(S) = 0$, it is straightforward to check that annihilators of elements in the essential left ideal $Re_{11} \oplus Re_{22} \oplus Re_{13}$ are not essential in $R$. Hence $Z(R) = 0$.

(c) If $\text{l.gl.dim} \ R \leq 1$, then

$$(0:e_{23}) = \begin{cases} 
a & b \\
0 & c & d \\
0 & 0 & e 
\end{cases} \quad c \in M; \quad a, b, d, e \in S/M.
$$

is generated by an idempotent element

$$\begin{pmatrix} 
u & 0 & v \\
0 & w & x \\
0 & 0 & y 
\end{pmatrix},$$

This forces $w^2 = w \in M$, which contradicts the hypothesis that $S$ contains no nontrivial idempotent elements.
Now let \( C \) be a left and right principal ideal domain with the following properties:
(a) \( C \) is a simple ring, which is not a division ring;
(b) every simple \( C \)-module is injective;
(c) there exists (up to isomorphism) only one simple \( C \)-module.

Examples of such rings have been provided by Cozzens [4]. Let \( M \) be a maximal left ideal of \( C \), and let \( I = I_C(M) = \{ x \in C \mid mx \in M \text{ for all } m \in M \} \) be the idealizer of \( M \) in \( C \) (see [6]). By [6, Theorem 4.3], \( I \) is a hereditary Noetherian integral domain with nontrivial, two-sided ideal \( M = M^2 \). By [6, Theorem 1.3 and Corollary 2.4], \( I \) has only two simple modules \( S_1 \) and \( S_2 \) (up to isomorphism); \( S_1 \) is a faithful injective simple module, \( S_0 \cong I/M \), and \( E(S_2)/S_2 \cong S_1 \) (where \( E(S_2) \) denotes the injective envelope of \( S_2 \)). By [8, Theorem 4], every nonzero singular \( I \)-module has a nonzero socle.

We now give a sequence of lemmas designed to show that \( I = S \) makes \( R \) the desired splitting ring.

**Lemma 2.** \( I \) is a splitting ring.

**Proof.** It is sufficient to show that \( \text{Ext}^1_I(F, T) = 0 \) for any nonsingular \( I \)-module \( F \) and any singular \( I \)-module \( T \).

As noted above, any singular \( I \)-module has nonzero socle; so \( \text{soc}(E(T)) = \text{soc} T \) is essential in \( E(T) \). Write \( \text{soc}(T) = X \oplus Y \), where every simple submodule of \( X \) is isomorphic to \( S_1 \) and every simple submodule of \( Y \) is isomorphic to \( S_2 \). Since \( S_1 \) is injective and \( I \) is Noetherian, then \( X \) is injective and \( E(T) \cong X \oplus E(Y) \). Moreover, either \( E(Y) = 0 \) or else \( E(Y) = E(\bigoplus_{\beta \in B} S_2^{(\beta)}) \cong \bigoplus_{\beta \in B} E(S_2^{(\beta)}) \) for some index set \( B \), where \( S_2^{(\beta)} \cong S_2 \) for all \( \beta \in B \). Hence \( B \neq \varnothing \) implies

\[
E(T)/\text{soc}(T) \cong \bigoplus_{\beta \in B} (E(S_2^{(\beta)})/S_2^{(\beta)}) \cong \bigoplus_{\beta \in B} S_1^{(\beta)}
\]

with \( S_1^{(\beta)} \cong S_1 \) for all \( \beta \in B \). Since \( I \) is hereditary and Noetherian, it follows that \( E(T)/\text{soc}(T) \) is injective. But \( T/\text{soc}(T) \) can be embedded (as a summand) in \( E(T)/\text{soc}(T) \); whence \( T/\text{soc}(T) \) is also injective. Hence we have the exact sequence

\[
\text{Ext}^1_I(F, \text{soc}(T)) \rightarrow \text{Ext}^1_I(F, T) \rightarrow \text{Ext}^1_I(F, T/\text{soc}(T)) = 0.
\]

Thus it is sufficient to show that \( \text{Ext}^1_I(F, \text{soc}(T)) = 0 \).

Since \( X \) is injective,

\[
\text{Ext}^1_I(F, \text{soc}(T)) \cong \text{Ext}^1_I(F, X \oplus Y)
\]

\[
\cong \text{Ext}^1_I(F, X) \oplus \text{Ext}^1_I(F, Y) \cong \text{Ext}^1_I(F, Y).
\]
From [5, Theorem 5.2], it follows that $F$ is a flat $I$-module. Since $I/M$ is a division ring, then $\text{Ext}_I^1(F, \text{soc}(T)) \cong \text{Ext}_I^1(F, Y) \cong \text{Ext}_I^1(I/M \otimes_I F, Y) = 0$ by [2, VI, Proposition 4.1.3]. Hence $I$ is a splitting ring.

Let
$$N = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \quad (d \in M; a, b, c, e, f \in I/M),$$
and let $U$ be the simple $R$-module $R/N$.

**Lemma 3.** $\text{Ext}_A^n(R, U) = 0$ for $n \geq 1$.

**Proof.** Since $A$ is a hereditary ring, $\text{Ext}_A^n(R, U) = 0$ for $n \geq 2$. As a left $A$-module, $R = \bigoplus \{ Ae_i \mid 1 \leq i \leq j \leq 3 \}$. Since $Ae_{11}, Ae_{12}, Ae_{13}, Ae_{22}, Ae_{33}$ are $A$-projective, then
$$\text{Ext}_A^1(Ae_i, U) \cong \bigoplus \{ \text{Ext}_A^1(Ae_i, U) \mid 1 \leq i \leq j \leq 3 \} \cong \text{Ext}_A^1(Ae_{23}, U).$$

Let $0 \to U \to X \to Ae_{23} \to 0$ be an exact sequence of $A$-modules. Since $e_{11}X = e_{11}U = 0$, $e_{11}X = 0$. Similarly $e_{33}X = 0$. Thus $X$ has identical $A$- and $I$-module structures, namely, $X \cong U \oplus Ae_{23}$ (since $E(S_2)/S_2 \cong S_1$ and $S_2 \cong U \cong Ae_{23}$ as $I$-modules). Therefore $\text{Ext}_A^1(Ae_{23}, U) = 0$.

**Lemma 4.** $\text{Hom}_A(R, U) \cong U \oplus U$.

**Proof.** $Ae_{11}, Ae_{12}, Ae_{13},$ and $Ae_{23}$ are nonsingular simple $A$-modules, and $U$ is a singular simple $A$-module. Therefore
$$\text{Hom}_A(Ae_{11} \oplus Ae_{12} \oplus Ae_{13} \oplus Ae_{23}, U) = 0.$$ Since $Ae_{23} \cong I$ and $Ae_{23} \cong U$ each have identical $A$- and $I$-module structures and since $IU$ is annihilated by $M$, then
$$\text{Hom}_A(R, U) \cong \text{Hom}_A(Ae_{23} \oplus Ae_{23}, U) \cong \text{Hom}_I(I \oplus U, U) \cong \text{Hom}_I(I, U) \oplus \text{Hom}_I(U, U) \cong U \oplus U.$$

**Lemma 5.** $\text{Ext}_A^n(A, U \oplus U) \cong \text{Ext}_A^n(A, U)$ for all $R$-modules $A$ and for all $n \geq 1$.

**Proof.** By Lemma 3 and [2, VI, Proposition 4.1.4],
$$\text{Ext}_A^n(A, \text{Hom}_A(R, U)) \cong \text{Ext}_A^n(A, U)$$
for all $R$-modules $A$ and for all $n \geq 1$. Thus the conclusion follows from Lemma 4.

**Lemma 6.** $\text{Ext}_A^1(A, U) = 0$ for all nonsingular $R$-modules $A$. 

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Proof. Since $\Lambda \cong I/M \oplus I/M \oplus I$, it follows from Lemma 2 that $\Lambda$ is a splitting ring. Therefore, $A = Z(\Lambda A) \oplus F$ for any nonsingular $R$-module $A$, and $\text{Ext}^1_\Lambda(F, U) = 0$. So it suffices to show that $\text{Ext}^1_\Lambda(Z(A), U) = 0$.

If $0 \neq \ell \in Z(\Lambda A)$, then the $\Lambda$-annihilator of $\ell$ is

$$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : b \in M; a, c \in I/M \right\}.$$

For let $K$ denote the $R$-annihilator of $\ell$, and let $L$ be the set of elements of $I$ which appear in the second row and second column of some element of $K$. Since $0 \neq \ell \in Z(\Lambda A)$, $L$ is a nontrivial left ideal of $I$. If $m \in M$, then

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is essential in $R$. Since $R/K$ is a nonsingular $R$-module, this forces

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in K.$$

It is now easy to verify that the $\Lambda$-annihilator has the desired form.

Thus $Z(\Lambda A)$ is 0 or a direct sum of simple $R$-modules isomorphic to $U$. As it was noted in the proof of Lemma 3, $\text{Ext}^1_\Lambda(U, U) = 0$. Consequently $\text{Ext}^1_\Lambda(Z(\Lambda A), U) = 0$.

Lemma 7. $\text{Ext}^1_R(A, B) = 0$, where $A$ is any nonsingular $R$-module and $B$ is any $R$-module isomorphic to a direct sum of copies of $U$.

Proof. Let $B = \bigoplus_{x \in J} U_x$, where $U_x \cong U$. Consider the exact sequence

$$(*) \quad 0 \to \bigoplus_{a \in \mathcal{I}} U_x \xrightarrow{f} \prod_{a \in \mathcal{I}} U_x \xrightarrow{\prod_{a \in \mathcal{I}}} \bigoplus_{a \in \mathcal{I}} U_x \to 0.$$

By Lemmas 5 and 6, $\text{Ext}^1_R(A, \prod_{x \in \mathcal{I}} U_x) \cong \prod_{x \in \mathcal{I}} \text{Ext}^1_R(A, U_x) = 0$. Thus $\text{Ext}^1_R(A, \bigoplus_{x \in \mathcal{I}} U_x) = 0$. Thus

$$\text{Hom}_R\left(A, \prod_{x \in \mathcal{I}} U_x\right) \xrightarrow{f_*} \text{Hom}_R\left(A, \bigoplus_{x \in \mathcal{I}} U_x\right) \to 0.$$

Since $NU = 0$, then $\text{Ext}^1_R(A, \bigoplus_{x \in \mathcal{I}} U_x) = 0$.

Since $NU = 0$, then $\text{Ext}^1_R(A, \bigoplus_{x \in \mathcal{I}} U_x)$ is also an exact sequence of $R/N$-modules. Since
$R/N$ is a division ring, $(\ast)$ splits as $R/N$-modules and as $R$-modules. Therefore $f_*$ is an epimorphism, and hence $\text{Ext}_R^1(A, \bigoplus_{a \in I} U_a) = \text{Ext}_R^1(A, B) = 0$ by exactness.

**Lemma 8.** Every simple singular $R$-module $V \cong U$ is $R$-injective.

**Proof.** If $V$ is a simple singular $R$-module not isomorphic to $U$, then $V \cong R/T$, where $T$ is of one of the following types of maximal left ideals of $I$:

(a) $T = \begin{cases} (a & b & c) \\ 0 & d & e \\ 0 & 0 & f \end{cases} d \in L; a, b, c, e, f \in I/M,$

where $L$ is a maximal left ideal of $I$ distinct from $M$;

(b) $T = \begin{cases} (a & b & c) \\ 0 & d & e \\ 0 & 0 & 0 \end{cases} d \in I; a, b, c, e \in I/M.$

To establish that $V$ is injective, it is sufficient to show that for any essential left ideal $K$ of $R$ and for any diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & K \\
\downarrow & & \downarrow f \\
& & R \\
\downarrow g & & \\
& & V
\end{array}
$$

there is an $h : R \rightarrow V$ such that $hf = g$. The left ideal $K$ can be any one of the following types, where $H$ denotes a nonzero left ideal of $I$.

(A) $K = \begin{cases} (x & y & z) \\ 0 & u & 0 \\ 0 & 0 & 0 \end{cases} u \in H; x, y, z \in I/M.$

(B) $K = \begin{cases} (x & y & z) \\ 0 & u & v \\ 0 & 0 & 0 \end{cases} u \in H; x, y, z, v \in I/M.$

(C) $K = \begin{cases} (x & y & z) \\ 0 & u & v \\ 0 & 0 & w \end{cases} u \in H; x, y, z, v, w \in I/M.$

(D) $K = \begin{cases} (x & y & z) \\ 0 & u & v(u) \\ 0 & 0 & 0 \end{cases} u \in H; x, y, z, v(u) \in I/M; \quad \text{if } u = 0, \text{ then } v(u) = 0.$
If $K$ is of type (D), the situation can be reduced to type (C) by the following argument. Let

$$P = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix} \quad a, b, c \in I/M.$$ 

Then

$$\begin{pmatrix} x & y & z \\ 0 & u & s \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} x & y & z \\ 0 & u & v(u) \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s - v(u) \\ 0 & 0 & 0 \end{pmatrix} \in K + P.$$ 

Therefore $K + P$ is of type (C). Since $Z(RV) = V$ and since $\text{soc}(K)$ is non-singular by Lemma 1(b), then $K \cap P \subseteq \text{soc}(K) \subseteq \ker g$. Hence $g$ can be extended to $K + P$ by setting $g(P) = 0$. Therefore the situation is reduced to type (C).

Let $T$ be of type (a). If $K$ is of type (A), (B) or (C), then $g$ can be extended to all of $R$ by using the fact that $V$ is an injective $R$-module.

If $F$ is of type (b) and $K$ is of type (A) or (B), then $e_{33}K = 0$ and $e_{33}V \neq 0$. Thus $g(K) = 0$, and the zero map extends $g$.

Finally, if $T$ is of type (b) and $K$ is of type (C), then $g$ can be extended to $h: R \to V$ by setting $h(x) = g(e_{33}x)$ for each $x \in R$.

**Theorem 9.** $R$ is a splitting ring with $\text{l.gl.dim} \ R = 2$.

**Proof.** To show $R$ is a splitting ring, it is sufficient to show that $\text{Ext}^1_R(F, T) = 0$ for any nonsingular $R$-module $F$ and for any singular $R$-module $T$. By Lemma 1(a), Lemma 8, and the fact that $N^2 = N$, $T/\text{soc}(T)$ is an injective, semisimple $R$-module. As in the proof of Lemma 2, it can be assumed that $\text{soc}(T)$ is a direct sum of copies of $U$. By Lemma 7, $\text{Ext}^1_R(F, \text{soc}(T)) = 0$. From the exact sequence

$$0 = \text{Ext}^1_R(F, \text{soc}(T)) \to \text{Ext}^1_R(F, T) \to \text{Ext}^1_R(F, T/\text{soc}(T)) = 0,$$

it follows that $\text{Ext}^1_R(F, T) = 0$. Therefore $R$ is a splitting ring.

Since any splitting ring $R$ satisfies $Z(R) = 0$ [3], then Lemma 1(c) and [7, Theorem 2.2] yield $\text{l.gl.dim} \ R = 2$.

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**References**


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