A NOTE ON JANKO’S SIMPLE GROUP
OF ORDER 175,560

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Abstract. Janko’s simple group J of order 175,560 is characterized among simple groups by the weak closure W of the involution in its centralizer. Among arbitrary finite groups, the theorem asserts that the normal closure of W is J.

1. Introduction. The existence of a simple group J of order 175,560 was established by Professor Z. Janko in [6]. Indeed, Janko proved that the group J was characterized by the presence of an involution t such that
(a) t lies in the center of a 2-Sylow subgroup of G,
(b) C_G(t) ∼ ⟨t⟩ × A_5,
(c) t is not central in G.

In this paper we prove the following:

Theorem. Let t be an involution in a group G. Assume (i) t lies in the center of a 2-Sylow subgroup of G and (ii) the weak closure of t in its centralizer in G has the form ⟨t⟩ × A_5. Then ⟨t^G⟩ is isomorphic to Janko’s simple group of order 175,560.

This "weak closure" version of a centralizer-characterization for J plays a key role in the study (being carried out by Professor M. Herzog and the author) of groups whose proper central 2-Sylow intersections are cyclic or generalized quaternion groups.

2. Proof of the theorem. The proof proceeds by a series of short steps. Let G be a minimal counterexample.

(1) t lies in no proper normal subgroup of G of 2-power index.
Assume t ∈ N ≤ G where G/N is a 2-group. Since by (i) |t^G| is odd, N transitively permutes the elements of t^G so

\[ t^G = t^N. \]

Since all conjugates of t lie in N, the weak closure of t in C_G(t) relative to G is also the weak closure of t in C_N(t) relative to N. Thus (i) and (ii)
hold with $N$ in place of $G$. If $|N| < |G|$, we may imply induction on $|N|$ to obtain

$$\langle t^G \rangle = \langle t^N \rangle \simeq J,$$

where $J$ denotes Janko's group. Since this presents us with the conclusion of the lemma, we may assume $|N| = |G|$. This allows us to assume (1).

(2) Set $W_t = \langle g^{-1}tg | g^{-1}tg \in C_G(t), g \in G \rangle$, the weak closure of $t$ in $C_G(t)$. Let $T$ be a fixed 2-Sylow subgroup of $W_t$. Then, by (ii), $T$ is elementary of order 8. $T$ is weakly closed in any 2-Sylow subgroup which contains $T$. Also $N(T)$ controls fusion in $T$.

Let $S$ be a 2-Sylow subgroup of $C(t)$. Then $S \cap W_t$ is a 2-Sylow subgroup of $W_t$ and so without loss of generality we may assume that $S \cap W_t = T$. Since $W_t$ is generated by conjugates of $t$ there exist a conjugate of $t$, say $t^g \in W_t - \langle t \rangle$. Then, by conjugating by elements in $W_t$ we may assume $t^g \in T$. Then, by conjugating by elements in $N_{W_t}(T)$ we see that conjugates of $t$ generate $T$, whence

$$(2.1) \quad S \cap W_t = T = \langle t^G \cap S \rangle.$$ 

Thus $T$ is the weak closure of $t$ in $S$ relative to $G$. This forces $T$ to be weakly closed in $S$—i.e., $T$ is the unique conjugate of $T$ lying in $S$. It follows that any 2-Sylow subgroup of $G$ contains only one conjugate of $T$. Thus $T$ is weakly closed in any 2-Sylow subgroup containing it. It follows from Sylow’s theorem that $T$ is weakly closed in any 2-subgroup of $G$ containing $T$.

If $a$ and $a^g$ both lie in $T$, then $T$ and $T^{g^{-1}}$ lie in $C(a)$. Then there exists an element $c \in C(a)$ such that $T^{g^{-1}c}$ and $T$ lie in a common 2-Sylow subgroup of $C(a)$. From the last line of the previous paragraph $T^{g^{-1}c} = T$. Thus $g^{-1}c \in N(T)$ and so $c^{-1}g \in N(T)$. Then $a^{c^{-1}g} = a^g$ and so the fusion $a \rightarrow a^g$ can be achieved in $N(T)$.

All assertions in (2) have been proved.

(3) $T^g$ is fused in $G$ (and in $N(T)$).

Since $S$ is a 2-Sylow subgroup of $N(T)$ and fusion in $T$ occurs in $N(T)$ we have $|t^G \cap T|$ is odd. Now in $W_t \cap N_G(T)$, there exists an element of order 3 normalizing $T$, and stabilizing and acting fixed point free on a subgroup of $T$ complementing $\langle t \rangle$. Indeed, $W_t \cap N_G(T)$ acts on $T$ with orbits of lengths 1, 3 and 3, and $t$ belongs to the orbit of length one. Since $T$ is the weak closure of $t$ in $S$, $t^G \cap S = t^G \cap T$ is a union of the $W_t \cap N_G(T)$-orbits mentioned above, has an odd number of elements and contains more than one element. It follows that $t^G \cap T = T^g$ and (3) is proved.

(4) $T$ lies in the center of every 2-Sylow subgroup containing it.

Since $[W_t, W_t] \simeq A_5$ is a normal subgroup of $C(t)$, we see that $C(t) \cap N(T)$ contains $[W_t, W_t] \cap N(T) \simeq A_4$ as a normal subgroup. Thus
$N(T)/C(T)$ is isomorphic to a subgroup of $SL(3, 2)$ which is (a) transitive on the seven elements of $T^#$ and (b) in which the subgroup $(N(T) \cap C(t))/C(T)$ fixing one of the elements of $T$, lies in the normalizer of

$$\{[W_t, W_t] \cap N(T)C(T) | C(T) \cong Z_3\},$$

corresponding to a 3-Sylow subgroup of $SL(3, 2)$. It follows that $[N(T):C(T)]=21$ or 42. Because the 7-Sylow normalizers of $SL(3, 2)$ are maximal in $SL(3, 2)$ we see that $N(T)/C(T)$ is the nonabelian group of order 21. Thus a 2-Sylow subgroup of $G$ lying in $N(T)$ lies in $C(T)$. Since $T$ is weakly closed in any 2-Sylow subgroup of $G$ containing it, it follows that $T$ lies in the center of every 2-Sylow subgroup containing it.

(5) Let $G = G(t^G)$ be the graph whose vertices are $t^G$, and whose arcs are commuting pairs of involutions in $t^G$. Let $G_t$ denote the connected component of $G$ containing $t$. Every element of odd order in $C(T)$ fixes $G_t$ pointwise.

Let $u$ denote an element of odd order in $C(T)$. Since $A_S$ admits no automorphism of odd order fixing pointwise one of its 2-Sylow subgroups, we see that for each $S \in T^#$, $u$ normalizes $W_s = \text{Vec}_T(s, C_G(s)) \cong \langle s \rangle \times A_S$ and hence centralizes each $W_s$. What this means is that if $T^s \cap T$ is non-trivial then $y$ also centralizes $T^s$. Since every commuting pair of involutions in $t^G$ lies in a conjugate of $T$, we have that $y$ centralizes every involution belonging to $G_t$.

(6) $O_{2^c}(G) = 1$.

It is easy to see that hypotheses (i) and (ii) inherit to $G/O_{2^c}(G)$. By induction, if $O_{2^c}(G) \neq 1$, $\langle t^G/O_{2^c}(G) \rangle \preceq J$. If $U = [C(t) \cap O_{2^c}(G), T]$, then $U$ is a subgroup of $W_t$ having odd orders and is normalized by $T$, a 2-Sylow subgroup of $W_t$. From the isomorphism type of $W_t$, $U = 1$. Then (using (3)), $C(t_1) \cap O_{2^c}(G) = C(T) \cap O_{2^c}(G)$ for all involutions $t_1$ in $T^#$. Since $T$ is noncyclic, $(C(t_1) \cap O_{2^c}(G)) | t_1 \in T^# = O_{2^c}(G)$ and so the previous sentence implies $O_{2^c}(G) \leq C(T)$. Thus $O_{2^c}(G)$ is centralized by $\langle t^G \rangle$. It follows that $\langle t^G \rangle$ is a perfect central extension of $J$ by a central group $Z$ of odd order. But since every odd Sylow subgroup of $J$ is cyclic and has a fixed-point-free element normalizing it, $J$ has no multipliers of odd order. Thus $\langle t^G \rangle$ splits over $Z$. But since it is generated by involutions and $|Z|$ is odd, it follows that $\langle t^G \rangle \cong J$, our desired conclusion. Thus we may assume $O_{2^c}(G) = 1$.

(7) Any two elements of 2-power order in $N(T)$ which are conjugate in $G$ are conjugate in $N(T)$.

Suppose $x$ and $g^{-1}xg = x^g$ are two elements of 2-power order in $N(T)$. By (4), $x$ and $x^g$ lie in $C(T)$. Then $T$ and $T^g$ lie in $C(x)$ and so $T^{g^{-1}c}$ and $T$ lie in a common 2-Sylow subgroup of $C(x)$ for an appropriate choice of $c$. Since by (2), $T$ is weakly closed in any 2-group containing it,
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$T^g = T$ so $g^{-1}c \in N(T)$. Then $c^{-1}g \in N(T)$ and $x^g = x^{g^{-1}}$ is conjugate to $x$ by an element in $N(T)$.

(8) $\mathcal{C}$ is not connected.

Suppose by way of contradiction that $\mathcal{C}$ is connected, so that $\mathcal{C} = \mathcal{C}_1$. Let $K$ be the centralizer in $G$ of $t^G$, so

$$K = \bigcap C(s), \quad s \text{ ranging over } t^G .$$

Our first objective will be to show that $K$ is trivial.

Set $K_0 = K \cap \langle t^G \rangle$. Then $K_0$ coincides with center of $\langle t^G \rangle$ and has 2-power order since $Z(\langle t^G \rangle)$ is necessarily a 2-group by (6).

Suppose $g$ is an element in $G$ such that conjugation by $g$ leaves the coset $tK_0$ fixed. Then $t^g = tk$ where $k \in K_0$. Since $[t, k] = 1$ and $t^g$ is an involution, either $k = 1$ or $k$ is an involution. In any event $k \in T$ since by (2) and (3), $T$ is the weak closure of $\langle t \rangle$ in $S$ where $S$ is any 2-Sylow subgroup of $G$ containing $T$. Assume $k \neq 1$. Again by (3), $T^g$ is fused so $k \in tG$. Then $k$ is a member of $tG$ commuting with all other members of $tG$. Since $G$ acts transitively on $t^G$, this means that all members of $tG$ are mutually commuting. Then $\langle t^G \rangle$ is elementary abelian. But then, on the other hand,

$$\langle t^G \rangle = \langle g^{-1}tg \mid g \in G, \ g^{-1}tg \in C_1(t) \rangle = W_t \cong Z_2 \times A_5,$$

a contradiction. Thus $k = 1$. Indeed, we have proved two things:

(2.2) \[ t^G \bigcap K_0 = \emptyset, \]

(2.4) \[ C_{t^G/K_0}(tK_0) = C_{t^G/K_0}. \]

Because of (2.3) and (2.4), hypotheses (i) and (ii) hold for $G/K_0$. Thus if $K_0 \neq 1$, induction on $G/K_0$ yields the fact that $\langle t^G \rangle$ is a perfect central extension of $J$ by $K_0$. But then $J$ has a perfect central extension by $K_0/K_0 \cong Z_2$. But in that case, $TK_0/K_0$ is a 2-Sylow subgroup of $\langle t^G \rangle/K_0 \cong J$, and $TK_0/K_0$ is elementary of order 8 and admits an automorphism of order 7. It follows that every coset of $K_0/K_0 \cong Z_2$ in $TK_0/K_0$ consists entirely of involutions. Thus $TK_0/K_0$ is elementary and so the 2-Sylow subgroups of $\langle t^G \rangle/K_0$ split over $K_0/K_0$. By a well-known theorem of Gaschütz [2] this implies that $\langle t^G \rangle/K_0$ splits over $K_0/K_0$, contrary to the fact that $\langle t^G \rangle/K_0$ is a perfect group. Thus we must assume

(2.5) \[ K_0 = 1. \]

Thus

(2.6) \[ \text{the centralizer of } tK/K \text{ in } G/K \text{ is covered by } C(t). \]

Also

(2.7) \[ t^G \bigcap K \text{ is empty} . \]
is an immediate consequence of (2.3). Now by (2.6) and (2.7), hypotheses (i) and (ii) hold for $G/K$. Then if $K \neq 1$, induction yields that $\langle t^G \rangle K/K \cong J$ and so $\langle t^G \rangle$ is a central extension of $J$ by $K \cap \langle t^G \rangle$. But now (2.5) implies $\langle t^G \rangle \cong J$, our conclusion. Thus we must assume

$$K = 1.$$  

(2.8)

Now by (5), (2.8) implies that $C(T)$ is a 2-Sylow subgroup of $G$. Then $N(T) = SB$ where $B$ is metacyclic of order $21$. Let $B_1$ denote a 3-Sylow subgroup of $B$. Then $B_1$ is a 3-Sylow subgroup of $W_t \cap N(T) \cong Z_2 \times A_5$ for some involution $s$ in $T^e$. Then since $S \leq N(W_s)$, $[S, B_1] \leq BT$. Since $B$ is generated by its 3-Sylow subgroups $[S, B] \leq BT$, also. Since $B$ normalizes $C(T) = S$, we have $[S, B] \leq BT \cap S = T$. Since $B$ and $S$ have coprime orders, $C_S(BT) = S$. Now $B$ acts without fixed points on $T$ so $C_S(BT) \cap BT \leq C_S(B) \cap (BT \cap S) = C_S(B) \cap T = \langle 1 \rangle$. Since $T$ is central in $S$, $C_S(B) = C_S(BT)$. Thus we have

$$C_S(BT) \times BT = N(T).$$

(2.9)

Suppose $C_S(BT) \neq 1$. Then $N(T)$ has a nontrivial 2-factor whose associated kernel contains $t$. Since $N(T)$ controls its fusion of 2-elements by (7), the focal subgroup of $S$ is proper in $S$ and contains $t$. It follows from the focal subgroup theorem [5] (see Theorems 3.4 and 3.5 of [4]) that $G$ contains a proper normal subgroup $N$ of 2-power index and $N$ contains $t$. But this contradicts step (1). Thus $C_S(BT) = 1$ and we now have

$$N(T) = BT, \quad T \text{ is a 2-Sylow subgroup of } G.$$

A Frattini argument now yields $C(t) = (N(T) \cap C(t))W_t$. But $N(T) \cap C(t)$ has the form $TB_1$ where $B_1$ is an appropriate 3-Sylow subgroup of $B$, and $TB_1 \leq W_t$. Thus

$$C(t) = W_t \cong Z_2 \times A_5.$$  

(2.11)

That $\langle t^G \rangle \cong J$ follows from (2.11) and (i) is a theorem of Janko [6]. Thus, on the assumption that $G$ is connected we reach our desired conclusion. Thus we may assume that $G$ is not connected, which is (8).

(9) Let $H_1$ be the stabilizer in $G$ of the set $C_1$. Then $H_1$ is a proper subgroup of $G$ and has the form $H_1 \cong Y \times J$, where $t^G \cap H_1 \leq J$. The centralizer (in $G$) of any involution in $Y \cap J$ lies in $H_1$.

Clearly for any $t_1 \in C_1$, $C(t_1) \leq H_1$. By (4), $H_1$ satisfies hypotheses (i) and (ii). Since $G$ is not connected by (8) and $G$ is transitive on the vertices of $C_1$ (since these are $t^G$), it follows that $H_1$ is a proper subgroup of $G$. Then induction on $|H_1|$ yields $\langle t^H \rangle \cong J$. Since $t^g \in C_1$ implies $g \in H_1$ (since the connected components $C_1$, ··· form a system of imprimitivity on $C_1$), we have

$$t^G \cap H_1 = t^H_1.$$  

(2.12)
Now $<t^{H_1}>$ is a normal subgroup of $H_1$ isomorphic to $J$. Since $J$ is a complete and simple group,

$$H_1 = C_{H_1}(<t^{H_1}> \times <t^{H_1}> \simeq Y \times J$$

where $Y \simeq C_{H_1}(<t^{H_1}>)$.

Since an involution in $J$ belongs to $\mathcal{C}_1$, we have already seen that the centralizer of any involution in $J$ lies in $H_1$.

Now let $x$ be an involution in $J$. Suppose, for some $g \in G$, $t^x = t^u$ where $u \in Y$. By (2.12), $u \in t^G \cap H_1 = t^{H_1}J \subseteq J$. Then $u \in Y \cap J = \{1\}$. Since $t^G \cap Y = \emptyset$ this means that for any element $w \in Y$, $C_G(w)$ contains $W_t$ and $C_G(w)/<w>$ satisfies hypotheses (i) and (ii). In particular, induction yields $<t^{C_t(x)}>/(<x>/Xt) \simeq J$. But since $<t^{H_1}>$ lies in $C(x)$ and is isomorphic to $J$, it follows that $<t^{C_t(x)}> \leq H_1$. Thus $C(x)$ stabilizes $t^{C_t(x)} = t^{H_1} = \mathcal{C}_1$, whence $C(x) \leq H_1$.

(10) Any involution $y \in H_1 - (Y \cup J)$ satisfies $C(y) \leq H_1$.

Let $y$ be an involution in $H_1 - (Y \cup J)$ so $y = y_1 y_2$ where $y_1$ is an involution in $Y$ and $y_2$ is an involution in $J$. Then $\mathcal{C}_1 \cap C(y)$ consists of 31 involutions distributed in $W_{t_2}$-orbits of lengths 1, 15 and 15 with representatives $t_1 = y_2$, $t_2$ and $t_3$, respectively. We see that $\mathcal{C}_1 \cap C(y) \simeq Z_2 \times A_5$ and without loss of generality we may assume $t_2$ lies in the $A_5$-part—i.e. $t_2 \in [W_{t_1}, W_{t_2}]$. If $t_3$ were conjugate to $t_1$ or $t_2$ in $C(y)$, then this fusion must occur in $H_1$ since the connected components of $\mathcal{C}$ form a system of imprimitivity. But since $C(y) \cap H_1$ has the form $C_2(y_1) \times C_f(t_1)$ this fusion cannot take place. We thus see that

$$(2.13) \quad <t_3^{C_t(y)} \cap H_1> \simeq A_5.$$
group having an elementary 2-Sylow subgroup (the centers entering into the central product having odd order). But $T_1$ must be a 2-Sylow center of this normal product. Also, $\langle t_3^{C(v)} \rangle^{H_1} \simeq A_5$ is a subgroup of $\langle t_3^{C(v)} \rangle$. It follows that either

(2.14) $\langle t_3^{C(v)} \rangle \simeq A_5$

or

(2.15) $\langle t_3^{C(v)} \rangle \simeq U(3, 4)^*$

where the asterisk indicates a perfect central extension of $U(3, 4)$ by a group of odd order (necessarily a 3-group). But in that case, $\langle t_3^{C(v)} \rangle^{H_1}$ corresponds to a subgroup of $U(3, 4)^*$ isomorphic to $A_5$, no two conjugates of which share a common involution (since the involutions of each lie in distinct connected components of $G$). Since this is clearly impossible, (2.14) holds. Then $t_3^{C(v)} \leq H_1$ and is stabilized by $C(y)$. It follows that $C(y) \leq H_1$.

(11) $H_1$ is strongly embedded in $G$.

This follows at once since $H_1$ contains a 2-Sylow normalizer and the centralizer of each of its involutions by (9) and (10).

A contradiction is now apparent. Since $H_1$ is a proper strongly embedded subgroup of $G$, by Bender’s theorem [1], $G$ contains exactly one nonabelian simple composition factor $F \simeq SL(2, q)$, $Sz(q)$ or $U(3, q)$ for $q$ a power of 2. But since $J$ is a nonabelian simple subgroup of $G$, $J$ is isomorphic to a subgroup of $F$. This is impossible since in the “Bender groups” centralizers of involutions are 2-closed while this subgroup-hereditary property fails in $J$.

**References**


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