A NOTE ON SYSTEMS OF LINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. Assume the existence and boundedness of a solution to a linear system of integrodifferential equations. Conditions are found which guarantee the solution is asymptotically almost periodic.

1. Introduction. We consider the asymptotic behavior as $t \to \infty$ of the bounded solutions of linear systems of equations of the form

$$\begin{align*}
(1) & \quad \int_{-\infty}^{\infty} x(t - \xi) \, dA_0(\xi) = f(t) \quad (-\infty < t < \infty), \\
(2) & \quad x^{(v)}(t) + \sum_{k=0}^{v-1} \int_{-\infty}^{\infty} x^{(k)}(t - \xi) \, dA_k(\xi) = f(t) \quad (-\infty < t < \infty).
\end{align*}$$

Here $f$ and $x$ are vectors with $N$ components and $A_k$ ($0 \leq k \leq v-1$) are $N$ by $N$ matrices. It is assumed that $A_k \in \text{NBV}(-\infty, \infty)$ for $0 \leq k \leq v-1$ (i.e. each component of $A_k$ is of bounded variation, left-continuous and vanishes at $-\infty$), and that

$$\begin{align*}
(1.1) & \quad \in L^\infty(-\infty, \infty), \quad \lim_{t \to \infty} f(t) = f(\infty) \text{ exists.}
\end{align*}$$

Let $\hat{A}(\lambda)$ denote the Fourier-Stieltjes transform

$$\hat{A}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda \xi} \, dA(\xi) \quad (-\infty < \lambda < \infty),$$

and define the spectral sets corresponding to (1) and (2) by

$$\begin{align*}
S_1 & = \{ \lambda \mid \hat{A}_0(\lambda) = 0, \quad -\infty < \lambda < \infty \}, \\
S_2 & = \{ \lambda \mid P(i\lambda) = 0, \quad -\infty < \lambda < \infty \},
\end{align*}$$

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where \( P(i\lambda) \) is the "characteristic function" associated with (2):

\[
P(i\lambda) = (i\lambda)^E + \sum_{k=0}^{v-1} (i\lambda)^k \hat{A}_k(\lambda) \quad (-\infty < \lambda < \infty),
\]

\[E = [\delta_{ij}], \text{ the } N \text{ by } N \text{ identity matrix.}\]

When the sets \( S_1 \) and \( S_2 \) are finite, Levin and Shea have shown that bounded solutions of (1) and (2) are almost periodic in a certain weak sense [2, Theorems 11a, 13]. They also give sufficient conditions which, for the scalar case \((N=1)\), guarantee that bounded solutions of (1) and, when \( v=1 \), of (2) are asymptotically almost periodic in the sense of Fréchet [1], [2, Theorems 5c, 5a]. The purpose of this note is to extend this latter result to systems of equations as well as to systems with higher order derivatives.

For each positive integer \( n \), consider the growth conditions.

\( H(f, n) \): \( f=(f_1, \cdots, f_N) \) satisfies (1.1),

\[
\int_0^\infty t^{n-1} |f_j(t) - f_j(\infty)| \, dt < \infty \quad (1 \leq j \leq N),
\]

\( H(A, n) \): \( A=[A_{ij}] \in NBV(-\infty, \infty), \)

\[
\int_{-\infty}^\infty |t|^n |dA_{ij}(t)| < \infty \quad (1 \leq i, j \leq N).
\]

**Theorem 1.** Let \( H(f, n) \) and \( H(A_0, n) \) hold, and suppose \( S_1=\{\lambda_1, \cdots, \lambda_n\}, \)

\[
(1.2) \quad (d/d(i\lambda))[\det \hat{A}_0(\lambda)] \neq 0 \quad (\lambda = \lambda_1, \cdots, \lambda_n).
\]

Let \( x(t) \) be a bounded, Borel measurable function which satisfies (1) on \((-\infty, \infty)\) as well as the tauberian condition

\( (T) \quad \lim_{t \to \infty, \tau \to 0} |x(t+\tau) - x(t)| = 0. \)

Then

\[
x(t) = f(\infty)A_0(\infty)^{-1} + \sum_{m=1}^{n} \gamma_m \exp[i\lambda_m t] + \eta(t) \quad (-\infty < t < \infty)
\]

where \( \gamma_m \in C^N \) \((1 \leq m \leq n)\) and \( \eta(t)\to 0 \) as \( t\to \infty \) (the term \( f(\infty)A_0(\infty)^{-1} \) does not appear in (1.3) if one of the \( \lambda_m=0 \)).

The analogous theorem for (2) is

**Theorem 2.** Let \( H(f, n), H(A_k, n) \) \((0 \leq k \leq v-1)\) hold, and assume \( S_k=\{\lambda_1, \cdots, \lambda_n\}, \)

\[
(1.4) \quad (d/d(i\lambda))[\det P(i\lambda)] \neq 0 \quad (\lambda = \lambda_1, \cdots, \lambda_n).
\]
Let \( x(t) \in L^\infty(-\infty, \infty) \) with \( x^{(\nu-1)}(t) \) locally absolutely continuous (LAC) on \((-\infty, \infty)\) satisfy (2) a.e. on \((-\infty, \infty)\). Suppose, in addition, that

\[
(1.5) \quad x^{(k)}(t) \in L^\infty(-\infty, \infty) \quad (1 \leq k \leq \nu - 1).
\]

Then (1.3) holds with \( \eta \) satisfying

\[
(1.6) \quad \lim_{t \to \infty} \eta^{(k)}(t) = 0 \quad (0 \leq k \leq \nu - 1), \quad \lim_{t \to \infty} \left( \text{ess sup} |\eta^{(\nu)}(t)| \right) = 0.
\]

We remark that the derivatives in (1.2) and (1.4) always exist whenever \( H(A_k, n) \) \((0 \leq k \leq \nu - 1)\) hold.

By a simple conversion process [2, Lemma 19.2] we may use Theorems 1 and 2 to obtain analogous results about the corresponding Volterra equations

\[
(1') \quad \int_0^t x(t - \xi) \, dA_k(\xi) = f(t) \quad (0 \leq t < \infty),
\]

\[
(2') \quad x^{(\nu)}(t) + \sum_{k=0}^{\nu-1} \int_0^t x^{(k)}(t - \xi) \, dA_k(\xi) = f(t) \quad (0 \leq t < \infty).
\]

In the case of (2'), we need not assume a priori (the analogue for \([0, \infty)\) of) hypothesis (1.5) since this will follow from the other hypotheses [2, Lemma 19.1].

**Corollary.** Let \( x(t) \in L^\infty(0, \infty) \) (with \( x^{(\nu-1)}(t) \in L^\text{LAC}[0, \infty) \)) satisfy (2') a.e. on \([0, \infty)\) with \( A_k = [A_k] \in NBV[0, \infty) \), and \( f \in L^\infty(0, \infty) \) such that \( \lim_{t \to \infty} f(t) = f(\infty) \) exists. Define \( S_2 \) and \( P(i\lambda) \) as before where the \( A_k(t) \) are understood to be identically zero on \((-\infty, 0)\). Suppose \( S_2 = \{\lambda_1, \ldots, \lambda_n\} \), and (1.4),

\[
(1.7) \quad \int_0^\infty t^n |dA_{ij}(t)| < \infty \quad (0 \leq k \leq \nu - 1, 1 \leq i, j \leq N),
\]

\[
(1.8) \quad \int_0^\infty t^{n-1} |f_j(t) - f_j(\infty)| \, dt < \infty \quad (1 \leq j \leq N)
\]

are satisfied. Then (1.3) holds on \([0, \infty)\) with \( \eta \) satisfying (1.6).

When \( n = 1 \), it is easy to see that (1.8) cannot be omitted from the hypotheses of this Corollary by observing that

\[
x(t) = \frac{1}{(\nu - 1)!} \int_0^t (1 - e^{-t})^{\nu-1} f(\tau) \, d\tau + x(0) \quad (0 \leq t < \infty)
\]

is a solution of the differential equation

\[
(1.9) \quad x^{(\nu)}(t) + \sum_{k=0}^{\nu-1} a_k x^{(k)}(t) = f(t) \quad (0 \leq t < \infty),
\]
where the constants $\alpha_k$ are chosen so that the characteristic polynomial of (1.9) is $P(z)=\prod_{k=0}^{n-1} (z+k)$. This is a special case of (2') with $N=1$, $A_k(0)=0$, $A_k(t)=\alpha_k$ ($t>0$, $0\leq k \leq n-1$).

2. **Proof of Theorem 1.** We may assume $N \geq 2$ since the case $N=1$ is [2, Theorem 5c]. Let $x \ast A$ denote the convolution

$$x \ast A(t) = \int_{-\infty}^{\infty} x(t - \xi) \, dA(\xi) \quad (-\infty < t < \infty),$$

that is $x \ast A=(z_1, \ldots, z_N)$ where $z_j(t) = \sum_{i=1}^{N} \int_{-\infty}^{\infty} x_i(t-\xi) \, dA_{ij}(\xi)$. By (1),

$$x \ast A \ast \text{adj} \ A(t) = f \ast \text{adj} \ A(t) \quad (-\infty < t < \infty),$$

where adj $A$ denotes the $N$ by $N$ matrix obtained by taking the formal adjoint of $A$, but with convolution replacing pointwise multiplication. This equation may be rewritten as $N$ scalar equations

$$(2.1) \quad x_j \ast B(t) = h_j(t) \quad (1 \leq j \leq N, \ -\infty < t < \infty),$$

where $B \in \text{NBV}(-\infty, \infty)$ is the scalar function defined by taking the formal determinant of $A$ in which pointwise multiplication is replaced by convolution, and $h=(h_1, \ldots, h_N)=f \ast \text{adj} \ A$. Using $H(A_0, n)$ and $H(f, n)$, one easily verifies that $H(B, n)$ and $H(h, n)$ hold with $\lim_{t \to \infty} h(t) = f(\infty) \text{adj} \ A_0(\infty)$. Since $B(\lambda) = \det[A_0(\lambda)]$ ($-\infty < \lambda < \infty$), the scalar case of Theorem 1 may be applied to each equation in (2.1) to yield

$$x_j(t) = h_j(\infty)B(\infty)^{-1} + \sum_{m=1}^{n} \gamma_{mj} \exp[i\lambda_m t] + \eta_j(t) \quad (1 \leq j \leq N, \ -\infty < t < \infty)$$

with $\gamma_{mj} \in C$ and $\eta_j(t) \to 0$ as $t \to \infty$ (the terms $h_j(\infty)B(\infty)^{-1}$ do not occur if one of the $\lambda_m = 0$). Theorem 1 follows by setting $\gamma_m=(\gamma_{m1}, \ldots, \gamma_{mN})$ for $1 \leq m \leq n$ and $\eta=(\eta_1, \ldots, \eta_N)$.

3. **Proof of Theorem 2.** We deduce Theorem 2 from Theorem 1. Let

$$G(t) = \int_{-\infty}^{t} \exp[-\xi^2] \, d\xi \quad (-\infty < t < \infty),$$

so that $H(G, n)$, $H(G', n)$ hold for any positive integer $n$, and

$$(3.1) \quad \hat{G}(\lambda) \neq 0, \quad (G(\lambda))^\prime = (i\lambda)\hat{G}(\lambda) \quad (-\infty < \lambda < \infty).$$

Using $x \ast a$ to denote $x \ast a(t) = \int_{-\infty}^{\infty} x(t-\xi) \, a(\xi) \, d\xi$, (2) gives

$$x_j^{(\ast)} \ast G'(t) + \sum_{k=0}^{m-1} \sum_{i=1}^{N} x_i^{(k)} \ast A_{kj} \ast G'(t) = f_j \ast G'(t) \quad (1 \leq j \leq N, \ -\infty < t < \infty)$$

with $\gamma_{mj} \in C$ and $f_j(\lambda) \to 0$ as $\lambda \to \infty$ (the terms $f_j(\infty) \text{adj} \ A_0(\infty)$ do not occur if one of the $\lambda_m = 0$). Theorem 1 follows by setting $\gamma_m=(\gamma_{m1}, \ldots, \gamma_{mN})$ for $1 \leq m \leq n$ and $\eta=(\eta_1, \ldots, \eta_N)$.
where $A_k = [A_{kij}]$. Integrating by parts yields

$$\dot{x}^{(v-1)}_j = G'(t) + \sum_{k=1}^{v-1} \sum_{i=1}^{N} x^{(k-1)}_i \star A_{kij} \star G'(t) + \sum_{i=1}^{N} x_i \star A_{0ij} \star G(t)$$

$$= f_j \star G'(t) \quad (1 \leq j \leq N, -\infty < t < \infty).$$

Repeating this convolution and integration process $v$ times, we obtain

(3.2) $x \star B(t) = g(t) \quad (-\infty < t < \infty)$

where $B = [B_{ij}]$ with

$$B_{ij} = \delta_{ij}(G')^v + \sum_{k=1}^{v-1} A_{kij} \star (G')^k \star G^{(v-k)} + A_{0ij} \star G^v,$$

$\delta_{ij}$ denoting the Kronecker delta, and $g = f \star C$ where $C = [C_{ij}]$ with $C_{ij} = \delta_{ij}(G')^v$. Here we have used $A^k$ and $G^v$ to denote the $k$-fold convolutions $A \star A \star \cdots \star A$, $a \star a \star \cdots \star a$. Using $H(G, n)$, $H(G', n)$, $H(A_k, n)$ ($0 \leq k \leq v-1$) and $H(f, n)$, one easily checks that $H(B, n)$ and $H(g, n)$ are satisfied with $\lim_{t \to -\infty} g(t) = f(\infty)((G(\infty))^v)$. The definition of $B$ and (3.1) imply

$$\det B(\lambda) = [G(\lambda)]^{(v)} \det[P(i\lambda)] \quad (-\infty < \lambda < \infty);$$

hence (2) and (3.2) have the same spectral sets. Also, $x$ satisfies (T) since $x \in LAC(-\infty, \infty)$ and $\|x'\|_{\infty} < \infty$. Thus, applying Theorem 1 to (3.2) and using the values of $g(\infty)$ and $B(\infty)$, we find (1.3) holds with $\gamma_m \in C^N (1 \leq m \leq n)$ and $\eta(t) \to 0$ as $t \to -\infty$. To show $\eta$ satisfies (1.6), note that (the analogue for $(-\infty, \infty)$ of) [2, Theorem 13a] implies

$$x(t) = f(\infty)A_0(\infty)^{-1} + \sum_{m=1}^{n} c_m(t) \exp[i\lambda_m t] + \eta_1(t) \quad (-\infty < t < \infty)$$

where $\eta_1$ satisfies (1.6), and the $c_m$ satisfy

$$c_m \in C^\infty(-\infty, \infty) \cap L^\infty(-\infty, \infty), \quad \lim_{t \to -\infty} c_m^{(j)}(t) = 0 \quad (j = 1, 2, \cdots).$$

Since $\sum_{m=1}^{n} (c_m(t) - \gamma_m) \exp[i\lambda_m t] \to 0$ as $t \to -\infty$, it follows from (3.3) that $c_m(t) \to \gamma_m$ as $t \to -\infty$ (1 $\leq m \leq n$). Thus $\eta$ satisfies (1.6).

We remark that the technique used to prove Theorem 2 may also be employed to give a different proof of Theorem 13a in [2].
References


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