HENSELIAN FIELDS AND SOLID $k$-VARIETIES. II
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ABSTRACT. Let $k$ be a real closed or Henselian field. A $k$-variety $X$ (affine) is said to be solid if $X$ is determined by its $k$ points. It is shown that a $k$-variety is solid if and only if it contains a nonsingular $k$ point. Another condition for solidity is given and a dimension theorem indicated.

0. Introduction. In [4] a solid $k$-variety is defined to be an affine $k$-variety which is determined by its $k$ points. For real closed and Henselian fields with absolute value, we gave a natural condition which is necessary and sufficient for a $k$-variety to be solid. This condition is here extended to any Henselian field. Moreover, we also demonstrate that for any real closed or Henselian field $k$, a necessary and sufficient condition for a $k$-variety to be solid is that the variety contain a nonsingular $k$ point. This condition is obviously insufficient when dealing with other fields such as the rationals. For example consider the curve $x^3+y^3=1$ defined over the rationals. It has only finitely many rational points all of which are nonsingular.

1. The projection condition. Let $A$ be a local integral domain with maximal ideal $m$. Let $A$, $m$ be Henselian and let $k$ be the quotient field of $A$. Then we wish to show that if $f(x)$ in $k[x]$ has a root in $k$, then so do certain nearby polynomials.

Lemma 1.1 (like [2, Lemma 5.10]). If $A$, $m$ is Henselian, $f(x) \in A[x]$, $\alpha \in A$ so that

(1) $f'(\alpha) = \mu$;

(2) for some $\delta \in m$, $f(\alpha) \equiv 0 \mod \mu^2 \delta$;

then $f(x)$ has a root $\alpha' \equiv \alpha \mod \mu \delta$.

Proof [2]. Let $z$ be a new variable and try to solve $f(\alpha + \mu z) = 0$ for $z$. First expand $f(\alpha + \mu z) = f(\alpha) + \mu f'(\alpha) + \mu^2 w(z)$ where $w(z) \in A[z]$ and is of degree $\geq 2$. Also $f(x) = \mu^2 \delta \delta'$ for some $\delta' \in A$. We want to solve

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362
Let \( h(z) = \delta z' + z + w(z) \) for \( z \). Let \( h(z) = \delta z' + z + w(z) \). Then \( h(0) = \delta z' \in m \), \( h'(0) = 1 \). But \( A \) is Henselian so \( h \) has a root \( z_0 \in m \). Then \( f(x + mu_0) = 0 \).

**Lemma 1.2.** Let \( h(x) \in A[x] \) be monic of degree \( r \) and suppose

1. \( h(\beta) = 0, \beta \in A \);
2. \( h'(\beta) = \mu \neq 0 \).

Then for \( \delta \in m \), \( g(x) \in A[x] \), \( f(x) = h(x) + h^2g(x) \) has a root in \( A \).

**Proof.** Note \( f'(\beta) = h'(\beta) + h^2g'(\beta) = \mu u \) for some unit \( u \in A \). But \( f(\beta) = h(\beta) + h^2g(\beta) = \mu h(\beta) \) and is thus divisible by \( \mu^2 \delta \). Now apply Lemma 1.1.

**Lemma 1.3.** Let \( f(x) \in k[x] \) be of degree \( r \) and let \( f(x) \) have a simple root \( \alpha \) in \( k \). Then there exists \( \gamma \in A \) so that for all \( g(x) \in A[x] \) of degree \( r \), \( f(x) + \gamma g(x) \) has a root in \( k \).

**Proof.** There exists \( b \in A \) so that \( b\alpha = \beta \in A \). Next define \( h(x) \) so that \( h(bx) = b \cdot f(x) \). Then one checks that \( h(x) \in A[x] \) and \( h(\beta) = 0 \). Next \( h'(\beta) = \mu \neq 0 \) since \( \alpha \) is a simple root of \( f \). By Lemma 1.2, if \( \delta \in m \), \( h(x) + \mu^2 h^2g(x) \) has a root \( \beta' \in A \) for any \( g(x) \in A[x] \). But then \( h(bx) + \mu^2 h^2g(bx) \) also has a root \( \beta' \mid b \in k \). So \( f(x) + b^{-r} \mu^2 h^2g(bx) \) also has a root \( \beta' \mid b \in k \). But this is sufficient for our purpose since if \( g(x) \in A[x] \) is of degree \( r \), then \( f^r(b^{-1} x) = 0 \) in \( A[x] \). So if \( \gamma = \mu^2 \delta \) then \( f(x) + b^{-r} \mu^2 \delta (b^r g(b^{-1} x)) = b(x) + \mu^2 h^2g(x) = f(x) + \gamma g(x) \) has a root in \( k \).

**Definition 1.3.** Given \( A \) and \( k \) as above, a point \( P \in k^d \) and \( \lambda \in A \), we wish to define a neighborhood of \( P \) in \( k^d \) which we call a \( \lambda \)-sphere. Namely, let \( n \) be an integer \( \geq 0 \) and let

\[ S_{\lambda, n} = \{ Q \in k^d | Q = P + \lambda v, \text{ where } v \in (m^n)^{\times d} \}. \]

**Definition 1.4.** Let \( k \) be a field, \( \bar{k} \) its algebraic closure. Let \( X \) be an affine \( k \)-variety which we consider as a subset of \( \bar{k}^n \) for some \( n \). Let \( X_k = X \cap k^n \). We say \( X \) is **solid** if \( I(X_k) = I(X) \) in \( k[X_1, \ldots, X_n] \). By \( I(X_k) \) we mean all polynomials in \( k[X_1, \ldots, X_n] \) which vanish on \( X_k \). Thus \( X \) is solid if \( X \) is determined by its \( k \) points.

We wish to give conditions on \( X \) which will be necessary and sufficient for \( X \) to be solid.

**Lemma 1.5.** Let \( X \) be a \( k \)-variety of dimension \( d \) and let \( \pi : X \to k^d \) be a morphism. Then \( X \) is solid if \( \pi_k(X_k) \supseteq S_{\lambda, n} \) for some \( \lambda \)-sphere \( S_{\lambda, n} \). 

**Proof.** It is easy to see that if \( f \in k[Y_1, \ldots, Y_d] \) vanishes on \( S_{\lambda, n} \), then \( f \equiv 0 \). If \( X \) is not solid, then \( X_k \subseteq W \) for some proper subalgebraic set \( W \) of \( X \). Then dimension \( W < d \) which implies \( \dim \pi(W) < d \). But \( \pi(W) \supseteq S_{\lambda, n} \) implies dimension \( \pi(W) \geq d \), a contradiction.
Let $X$ be a variety of dimension $d$. Then if $k[x_1, \ldots, x_n]$ is the coordinate ring of $X$, by Noether normalization [6, p. 266], we can assume $x_1, \ldots, x_d$ are independent transcendental and $k[x_1, \ldots, x_n]$ is integral and separable over $k[x_1, \ldots, x_d]$. Let $\pi: X \to k^d$ be the induced morphism.

**PROPOSITION 1.6.** Let $k$ be Henselian, i.e., the quotient field of a Henselian ring $A$. Let $X$ be a $k$-variety of dimension $d$ and $\pi: X \to k^d$ as above. Then $X$ is solid if and only if $\pi(X_k)$ contains a $\lambda$-sphere.

**Proof.** The proof is the same as that given in [4] except that here one gets a $\lambda$-sphere. First choose $z \in k[x_1, \ldots, x_n]$ so that the quotient field of $k[x_1, \ldots, x_d, z]=\text{quotient field of } k[x_1, \ldots, x_n]$. Then let $f(x_1, \ldots, x_d, z)=$ the primitive irreducible polynomial of $z$ over $k(x_1, \ldots, x_d)$. Then $f(x_1, \ldots, x_d, Z)=\sum_{i=0}^{m} a_i(x_1, \ldots, x_d)Z^i$ is irreducible in $k[x_1, \ldots, x_d, Z]$.

Now $x_{d+1}=\sum_{j=0}^{m} (b_{ij}(x_1, \ldots, x_d))z^j$ where $b_{ij}, c_{ij} \in k[x_1, \ldots, x_d]$. We let $U=\{P' \in X|a_m(P')\neq 0, \text{ all } c_{ij}(P')\neq 0 \text{ and } (\partial f/\partial z)(P')\neq 0\}$. Noting $U$ is nonempty, we can choose $P' \in U$. Let $P=\pi(P')$.

Now choose $\lambda$ so that if $Q \in S_{P, x_1}$ then $a_m(Q)\neq 0$, all $c_{ij}(Q)\neq 0$ and, using Lemma 1.3, $f(Q, Z)$ has a root $a \in k$.

Then we let $Q'=(x_1(Q), \ldots, x_d(Q), x_{d+1}(Q, a), \ldots, x_n(Q, a))$. And as in [4] show that $Q'$ is a $k$ point of $X$ and $\pi(Q')=Q$. This shows $\pi(X_k) \supset S_{P, x_1}$.

From Proposition 1.6 it is possible to prove as in [4] a dimension theorem.

**THEOREM 1.7.** Let $k$ be a Henselian field, $X$ a solid $k$-variety of dimension $d$. Let $W_1, \ldots, W_r$ be subvarieties of $X$ of dimension $\leq d-2$. Then there exists a solid $k$-variety $W$ contained in $X$ with dimension $W=d-1$ and $W=W_1 \cup \cdots \cup W_r$.

**Proof.** Just as in [4, Theorem 3].

2. The nonsingular point condition.

**THEOREM 2.1.** Let $k$ be Henselian or real closed. A $k$-variety $X$ is solid if and only if $X_k$ contains a nonsingular point of $X$.

**Proof.** First the Henselian case. Let $Q$ be a nonsingular point of $X$. Let $k=\text{the quotient field of } A$ and let $k[x_1, \ldots, x_n]$ be the polynomial ring. Let $d=\text{dim } X$. Then we can find $f_1, \ldots, f_r, r=n-d$, in $k[x_1, \ldots, x_n]$ so that $X=V(f_1, \ldots, f_r)$ in a neighborhood of $Q$. We can assume all $f_i \in A[x_1, \ldots, x_n]$. We know rank($(\partial f_i/\partial x_j)(Q))=r$. Then by reordering the $x_i$'s, we can assume

$$\det_{i,j=1, \ldots, r}((\partial f_i/\partial x_j)(Q)) = \mu \neq 0.$$
We next want to apply Lemma 5.10 of [2]. To do this, we need to change $Q=(a_1, \ldots, a_r, b_1, \ldots, b_d)=(a, b)$. There exists $\gamma \in A$ so that $\gamma a_i, \gamma b_j \in A$ for all $i, j$. Let $\gamma a=(\gamma a_1, \ldots, \gamma a_r)$. Let $d_i=$ degree of $f_i$. We then let $h_i(\gamma x)=\gamma^{d_i}f_i(x)$ and then $h_i(\gamma a, \gamma b)=0$. Moreover $(\partial h_i/\partial x_j)(\gamma a, \gamma b)=\gamma^{d_i-1}(\partial f_i/\partial x_j)(a, b)$ so we get $\det_{1 \leq i, j \leq r}(\partial h_i/\partial x_j)(\gamma a, \gamma b)=\gamma^s u$ for some $s$. Now choose $b'$ so $\gamma b'=\gamma b+\gamma^{2s}u^2$ where $v \in m^{\times d}$. Then $h_i(\gamma a, \gamma b') \equiv 0 \mod \mu^2\gamma^{2s}m$, all $i$. And

$$\det((\partial h_i/\partial x_j)(\gamma a, \gamma b')) \equiv \gamma^s u \mod \gamma^{2s}u^2m,$$

and so

$$= \gamma^s uu \quad \text{where } u \text{ is a unit in } A.$$

By Lemma 5.10 of [2], there exists $a' \in k$ so that $h_i(a', b')=0$ for all $i$. Then $f_i(a', b')=0$ for all $i$. This means $b' \in \pi(X_k)$ where $\pi(a, b)=b$. Letting $\lambda=\mu^2\gamma^{2s}$ and $P=\pi(Q)$, we have $\pi(X_k) \supset P_{\lambda, a, 1}$, and so by Lemma 1.5, $X$ is solid.

For the real closed case, we need to prove an implicit function theorem.

**Lemma 2.2.** Let $k$ be a real closed field and $f_1, \ldots, f_r \in k[x_1, \ldots, x_r]$. Let $X=V(f_1, \ldots, f_r)$. Let $P \in X_k$ and suppose $\det((\partial f_i/\partial x_j)(P)) \neq 0$, $i, j=1, \ldots, r$. Then there exists $\epsilon \neq 0$ in $k$ such that the following holds: Let $P=(a_1, \ldots, a_n)$, then if $\sum_{i=r+1}^{n}(b_i-a_i)^2 \leq \epsilon^2$, there exist $b_1, \ldots, b_r$ in $k$ such that $f_i(b_1, \ldots, b_r)=0$, $i=1, \ldots, r$. In other words, $\sum_{i=r+1}^{n}(b_i-a_i)^2 \leq \epsilon^2$ implies $(b_{r+1}, \ldots, b_n)$ in $\pi(X_k)$ where $\pi: X \to k^d$ is the obvious projection.

**Proof.** We apply the Tarski-Seidenberg criterion given in Jacobson [5, p. 314], which states:

Let $t=(t_1, \ldots, t_r)$, $x=(x_1, \ldots, x_n)$. Let $f \in Q[t, x]$, $Q$ the rational numbers. Then let $f(t, x)=0$ be an equality which has solutions for $x$ in $k^n$ for all substitutions for $t$ in $k^r$, for some real closed field $k$.

**Conclusion.** For every real closed field $k$, we have solutions for $x$ in $k^n$ for all substitutions for $t$ in $k^r$.

To translate our situation to the above, we must add new variables $t=(t_1, \ldots, t_r)$ as "dummy variables" to get polynomials $f(t, x) \in Q[t, x]$ so that substituting correctly for $t$ in $k^r$, we obtain the $f_i(x)$. Next let $g(t, x_1, \ldots, x_n)=\det((\partial f_i/\partial x_j), i, j=1, \ldots, r)$. Adding a new variable $x_{n+1}$, we consider the polynomial

$$f(t, x_1, \ldots, x_{n+1}) = \sum_{i=1}^{r} f_i^2(t, x) + (1 - x_{n+1}g(t, x))^2 \in Q[t, x].$$

Add new variables $y_{r+1}, \ldots, y_n, z, \epsilon$ and let

$$h(x_{r+1}, \ldots, x_n, y_{r+1}, \ldots, y_n, z, \epsilon) = \sum_{i=r+1}^{n} (x_i-y_i)^2 - \epsilon^2 + z^2.$$

Now note that the statement in Lemma 2.2 is equivalent to: For every substitution of $t_0$ for $t$ in $k^r$, if the equation $f(t_0, x_1, \ldots, x_{n+1})=0$ has a
solution for $x_1, \cdots, x_{n+1}$ in $k$, then there exists $\varepsilon \neq 0$ so that if $y_{r+1}, \cdots, y_n, z \in k$ and $h(x_{r+1}, \cdots, x_n, y_{r+1}, \cdots, y_n, z, \varepsilon) = 0$, there exists $y_1, \cdots, y_r \in k$ so that $f(t_0, y) = 0$. Add new variables $u, v$ and let

$$\alpha(t, x, y, z, u, v, \varepsilon) = (1 - uf(t, x))^2(1 - ev)^2 + (1 - h(x, y, z, \varepsilon)w)^2f^2(t, y).$$

Now one verifies that Lemma 2.2 for $k$ is equivalent to the statement that $\alpha(t, x, y, z, u, v, \varepsilon) = 0$ has a solution in the remaining variables for all choices of $t_1, \cdots, t_r, x_1, \cdots, x_{n+1}, y_{r+1}, \cdots, y_n, z$ in $k$. But Lemma 2.2 is true for $k = \mathbb{R}$, the real numbers, by the implicit function theorem [1, p. 147]. Thus, by the theorem in Jacobson [5] quoted at the start of the proof, Lemma 2.2 is true for all real closed fields $k$.

To apply Lemma 2.2 to prove the real part of Theorem 2.1, choose $P$ nonsingular in $X_k$. As in the Henselian case, we can find a neighborhood of $P$ where $X = V(f_1, \cdots, f_r), f_i \in k[x_1, \cdots, x_n], r = n - d, d = \dim X$ and $\det_{i,j=1,\ldots,r}(\partial f_i/\partial x_j)(P) \neq 0$.

By Lemma 2.2, $\pi(X_k)$ contains a sphere in $k^d$. Then by the real equivalent of Lemma 1.5 (see [3, Theorem 1]) we are done.

To see that a solid $k$-variety contains a nonsingular $k$ point, just note that the set $U$ of nonsingular points of $X$ is Zariski open $X$; and, since $X$ is solid, $X_k \cap U$ is not empty.

**Bibliography**


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