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# A PROBLEM CONNECTED WITH THE ZEROS OF RIEMANN'S ZETA FUNCTION

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ABSTRACT. Estimates are given for the number of zeros of $\mathrm{Re}\{\pi^{-s/2}\Gamma(s/2)\zeta(s)\}$ and $\mathrm{Im}\{\pi^{-s/2}\Gamma(s/2)\zeta(s)\}$ with $0<\mathrm{Im}\, s<T$, and fixed $\mathrm{Re}\, s$ inside the critical strip.

Let $f(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$, where $\zeta$ is Riemann's zeta function. Suppose that $\lambda$ is a fixed real number such that $\frac{1}{2}<\lambda<1$, and let

$$R(t) = \mathrm{Re}\, f(\lambda + it), \qquad I(t) = \mathrm{Im}\, f(\lambda + it).$$

Finally let $N_R(\lambda, T)$ and $N_I(\lambda, T)$ be the number of zeros of $R(t)$ and $I(t)$, respectively, in the interval $0<t<T$, multiple zeros being counted according to their multiplicity.

Berlowitz [1] proved that $N_R(\lambda, T)$ and $N_I(\lambda, T)$ are unbounded as $T\to\infty$. Berndt [2] improved on this result by showing that there exists a positive constant $A$ such that, for all sufficiently large $T$,

$$N_R(\lambda, T) > AT, \qquad N_I(\lambda, T) > AT.$$

The object of this paper is to obtain the following further improvement.

THEOREM.[1] *For every $\lambda$ such that $\frac{1}{2}<\lambda<1$, there exists a positive constant $A$ such that, for all sufficiently large $T$,*

$$N_R(\lambda, T) \geqq (1/2\pi)T\log T - AT, \qquad N_I(\lambda, T) \geqq (1/2\pi)T\log T - AT.$$

PROOF. We shall prove the Theorem for $N_R(\lambda, T)$; the proof for $N_I(\lambda, T)$ is the same, except for obvious changes of notation. Let $\frac{1}{2}<\lambda<1$, $T>0$. Without loss of generality we may assume that $T$ is not the imaginary part of a zero of $\zeta(s)$.

Let $L$ be the straight line segment joining the points $\lambda$ and $\lambda+iT$ modified, if need be, by small semicircular indentations so that any zeros of $\zeta(s)$ with real part $\lambda$ lie to the left of $L$. It is obvious that the number of

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[1] Substantially the same result was obtained by N. Levinson in a paper, *On theorems of Berlowitz and Berndt*, which appeared in J. Number Theory 3 (1971), 502–504, shortly after our paper had been submitted.

distinct zeros of $\operatorname{Re} f(s)$ on $L$ is not less than $[(1/\pi)\Delta_L \arg f(s)]$. If $\zeta(s)$, and hence $f(s)$, has a zero of order $n$ at $\lambda+it_0$ then $R(t)$ has a zero of the same or higher order at $t_0$; on the other hand, if $\gamma$ be the semicircular detour by which $L$ avoids $\lambda+it_0$, then $\operatorname{Re} f(s)$ has at most $n+1$ distinct zeros on $\gamma$, since

$$(d/d\theta)\arg f(\lambda + it_0 + \delta e^{i\theta}) = \operatorname{Im}\{(f'/f)(\lambda + it_0 + \delta e^{i\theta})i\delta e^{i\theta}\} = n + O(\delta)$$

so that $\arg f(s)$ increases monotonically by $n\pi+O(\delta)$ on $\gamma$, it being assumed that the radius $\delta$ of $\gamma$ is sufficiently small. Hence it follows easily that

$$N_R(\lambda, T) \geqq [(1/\pi)\Delta_L \arg f(s)] - N_0(\lambda, T), \tag{1}$$

where $N_0(\lambda, T)$ is the number of zeros of $\zeta(s)$ on the straight line segment from $\lambda$ to $\lambda+iT$. It is well known that $N_0(\lambda, T)=O(T)$, and it is easily shown (cf. the Lemma below) that

$$\Delta_L \arg f(s) = \tfrac{1}{2}T \log T + O(T). \tag{2}$$

By (1) and (2) the Theorem is thus established.

Lemma. *If $L$ is defined as in the proof of the Theorem above, then* (2) *holds.*

Proof. Let $L'$ be the contour which is symmetric to $L$ about the line of real part $\frac{1}{2}$; and let $C$ be the simple closed contour which goes along $L$ from $\lambda$ to $\lambda+iT$, then straight from $\lambda+iT$ to $1-\lambda+iT$, then along $L'$ from $1-\lambda+iT$ to $1-\lambda$, and finally straight from $1-\lambda$ back to $\lambda$.

It is well known that the number of zeros of $f(s)$ (or, equivalently, of $\zeta(s)$) inside $C$ is $(1/2\pi)T\log T+O(T)$. Hence

$$\Delta_C \arg f(s) = T \log T + O(T). \tag{3}$$

But $f(s)$ is real on the real axis, and $f(1-s)=f(s)$ and $f(\bar{s})=\overline{f(s)}$ for all $s$. Hence

$$\Delta_C \arg f(s) = 2\Delta_L \arg f(s) + 2\Delta_K \arg f(s), \tag{4}$$

where $K$ is the straight line segment from $\lambda+iT$ to $\frac{1}{2}+iT$. But

$$\Delta_K \arg(\pi^{-s/2}) = \Delta_K \operatorname{Im} \log(\pi^{-s/2}) = 0,$$
$$\Delta_K \arg \Gamma(s/2) = \Delta_K \operatorname{Im} \log \Gamma(s/2) = O(1),$$

as may be seen by Stirling's formula, and

$$\Delta_K \arg \zeta(s) = -\int_{1/2+iT}^{\lambda+iT} \operatorname{Im}(\zeta'(s)/\zeta(s))\, ds = O(\log T),$$

as may be shown by the method given in Davenport [**3**, pp. 103–104]. Hence

$$\Delta_K \arg f(s) = O(\log T). \tag{5}$$

The desired result (2) now follows from (3), (4), and (5).

By using improved estimates for $N_0(\lambda, T)$ and for the number of zeros of $\zeta(s)$ inside $C$ (see Titchmarch [**4**, Theorems 9.4 and 9.19]), it can easily be seen that the term $AT$ in our Theorem may be replaced by

$$(1/2\pi)(1 + \log(2\pi))T + o(T).$$

## REFERENCES

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