ON Q-DENSE AND DENSELY DIVISIBLE LCA GROUPS
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Abstract. In this note the structure of those LCA groups containing a dense subgroup isomorphic to the additive group of rational numbers is investigated. As a result of this investigation, it is shown that an LCA group containing a dense divisible subgroup of finite rank must be divisible.

In [1] the authors investigate the structure of those locally compact (Hausdorff) abelian (LCA) groups $G$ containing a dense subgroup of the form $Z(p^n)$, where $p$ is a prime. In this note we turn our attention to the following related (and more general) question: Which LCA groups contain as a dense subgroup a homomorphic image of the additive group $Q$ of rational numbers?

All groups considered will be LCA unless otherwise specified. The additive group of the rational numbers is denoted by $Q$; when considered as a topological group, it has the discrete topology. By $R$ we denote the group of real numbers with its usual topology. If $p$ is a prime we denote by $Z(p^n)$ the (discrete) group of $p^n$th complex roots of unity. The $p$-adic integers and $p$-adic numbers are denoted by $J_p$ and $F_p$ respectively, with their usual topologies defined as in [3, §10]. Let $G$ be compact and let $K$ be a cardinal number. We write $G^K$ for the (full) direct product of $K$ copies of $G$ with the compact product topology. If $G$ is discrete, we write $G^K$ for the (weak) direct sum of $K$ copies of $G$, taken discrete. The character group of $G$ is written $\hat{G}$.

Let $E$ denote the minimal divisible extension [3, A.15] of the LCA group $G$. We always topologize $E$ as in [3, 25.32(a)], so that $E$ becomes a divisible LCA group containing $G$ as an open subgroup. For example, if $G = \prod J_p$, by which we will always mean the full direct product of the groups $J_p$, one for each prime $p$, then $E$ is topologically isomorphic to the local direct product [3, 6.16] of the groups $F_p$ relative to the open subgroups $J_p$, one factor for each $p$ [2, 25.32(d)]. We write this group as $L^p(F_p)$. We shall use the fact that $L^p(F_p)$ is self-dual [3, 25.34(b)]. We now state a lemma for future reference.
Lemma 1. Let \( f \) be a continuous monomorphism from a torsion-free LCA group \( G \) into a divisible LCA group \( D \). Then \( f \) may be extended to a continuous monomorphism \( \tilde{f}: E \rightarrow D \), where \( E \) is the minimal divisible extension of \( G \).

Proof. The homomorphism \( f \) may be extended to a homomorphism \( \tilde{f}: E \rightarrow D \) by [3, A.7]. Continuity of \( \tilde{f} \) follows from the openness of \( G \) in \( E \). That \( \tilde{f} \) is one-one follows from the fact that \( E/G \) is a torsion group [3, A.17].

Definition. An LCA group \( G \) is called \( Q \)-dense if and only if there exists a homomorphism \( f: Q \rightarrow G \) having dense image.

We may reformulate this definition by using the adjoint map [3, 24.37]. Thus an LCA group \( G \) is \( Q \)-dense if and only if there exists a continuous monomorphism \( f^* \) mapping \( G \) into \( Q \) [3, 24.41]. Obvious examples of \( Q \)-dense groups are \( Q \) and \( R \). By using our second formulation just given, it is easy to show that a compact group is \( Q \)-dense if and only if it is solenoidal [3, 9.2]. (Here we use the fact that \( \hat{Q} \) is algebraically isomorphic to \( Q^\omega \), where \( \omega \) denotes the power of the continuum.) In particular, \( \hat{Q} \) is \( Q \)-dense. An example of a totally disconnected \( Q \)-dense group other than \( Q \) is \( F_p \), and more generally \( LP(F_p) \). In fact, the next lemma shows that the direct product of \( R \), \( \hat{Q} \), and \( LP(F_p) \) is also \( Q \)-dense. This lemma will lead us to a description of the class of \( Q \)-dense groups. The authors are indebted to the referee for suggesting an important simplification in its proof.

Lemma 2. The group \( R \times LP(F_p) \times \hat{Q} \) is \( Q \)-dense.

Proof. We must show that there exists a continuous monomorphism \( f \) from \( R \times LP(F_p) \times \hat{Q} \) into \( \hat{Q} \). First, we set \( A = R \times LP(F_p) \), the group of adèles (see [2, §3.1]). Like \( R \), each group \( F_p \) contains canonically a dense subgroup isomorphic to the rationals, which we denote by \( Q \) in each case. It is shown in [2, §3.6] that there exists a continuous homomorphism \( \pi \) from \( A \) onto \( \hat{Q} \) whose kernel is the subgroup of \( A \) consisting of all sequences of the form \((x, x, x, \cdots)\) with \( x \in Q \). Let \( \varphi: A \rightarrow Q \) be defined by \( \varphi(x_0, x_2, \cdots, x_p, \cdots) = (x_0, 2x_2, \cdots, px_p, \cdots) \). Clearly, \( \varphi \) is a continuous monomorphism. Moreover, we see that \( \varphi(A) \cap \ker \pi = \{0\} \); this follows from the fact that no sequence of the form \((n, n, n, \cdots)\), where \( n \) is an integer, can belong to \( \varphi(A) \), since \( n \in pJ_p \) for at most finitely many primes \( p \). Hence the map \( \psi = \pi \circ \varphi \) is a continuous monomorphism from \( A \) into \( \hat{Q} \).

In order to complete our construction of \( f \), it will be necessary to show that the quotient group \( \hat{Q}/\psi(A) \) has cardinal number \( c \). To do this, we first define a homomorphism \( \pi' \) mapping \( A/\psi(A) \) onto \( \hat{Q}/\psi(A) \) by the rule
\pi'(a + \varphi(A)) = \pi(a) + \varphi(A), \text{ where } a \in A. \text{ We then observe that } \ker \pi' \text{ is isomorphic to the countable group } Q, \text{ and so } \hat{Q}/\varphi(A) \text{ will have cardinality } c \text{ if } A/\varphi(A) \text{ does also. But the cardinality of } A/\varphi(A) \text{ certainly exceeds that of } \\
\prod J_{p}(\varphi(A) \cap \prod J_{p}) = [\prod J_{p}] [\prod pJ_{p}] > [\prod pJ_{p}] [\prod Z(h)]/H, \text{ where } \\
\prod Z(p) \text{ is the (uncountable) full direct product of the } p \text{-element cyclic groups } Z(p) \text{ and } H \text{ is countable. Hence } A/\varphi(A) \text{ has cardinality } c, \text{ and therefore so does } \hat{Q}/\varphi(A).

Finally, we observe that \varphi(A) is a divisible subgroup of \hat{Q}. Hence there exists a subgroup } D \text{ of } \hat{Q} \text{ such that algebraically, } \hat{Q} \cong \varphi(A) \times D, \text{ where } D \cong \hat{Q}/\varphi(A) \text{ [3, A.8]. Since } D \text{ must be divisible, it follows from [3, A.14] and the preceding paragraph that } D \cong Q^{\infty}. \text{ Combining this isomorphism with the monomorphism } \psi: A \rightarrow \varphi(A) \text{ we can easily construct the desired continuous monomorphism } f \text{ mapping } A \times Q^{\infty} \text{ into } \hat{Q}. \text{ This completes the proof.}

We are now able to give a description of the Q-dense LCA groups.

**Theorem 1.** An LCA group } G \text{ is Q-dense if and only if } G \text{ is topologically isomorphic to } Q \text{ or to a quotient of } R \times \text{LP}(F_{p}) \times Q^{\infty} \text{ by a closed subgroup. In particular, every Q-dense LCA group is divisible.}

**Proof.** If } G \text{ has the form mentioned, it follows immediately, from Lemma 2 and the fact that a continuous homomorphic image of a Q-dense group is Q-dense, that } G \text{ is Q-dense.}

For the converse, assume that } G \text{ is Q-dense. Then there exists a continuous monomorphism } f: G \rightarrow \hat{Q}. \text{ Now } \hat{G} \text{ can be written in the form } \hat{G} \cong R^{n} \times \hat{G}_{0}, \text{ where } n \text{ is a nonnegative integer and } \hat{G}_{0} \text{ has a compact open subgroup [3, 24.30]. Now we must have } n \leq 1, \text{ for if } n > 1 \text{ we could find a continuous monomorphism from } R^{2} \text{ into } \hat{Q}. \text{ This would imply the existence, via the adjoint map, of a continuous homomorphism from } Q \text{ into } R^{2} \text{ having dense image, which is absurd. Next, let } K \text{ be the (compact) identity component of } \hat{G}_{0}. \text{ If } K \neq \{0\}, \text{ then } f(K) \text{ is a nontrivial closed connected subgroup of } \hat{Q}, \text{ and hence } f(K) = Q \text{ (this follows from the fact that all proper quotients of } Q \text{ are torsion groups). Hence } \hat{G} = K \text{ and } f \text{ is a topological isomorphism, so } G \text{ is just the discrete group } Q. \text{ On the other hand, if } K = \{0\}, \text{ we have that } \hat{G}_{0} \text{ is a totally disconnected torsion-free group, so that its minimal divisible extension } E \text{ [3, 25.32] is also totally disconnected [3, 7.8] and torsion-free [3, A.16]. We conclude from [3, 25.33] that } E \text{ has the form } Q^{n} \times E_{0}, \text{ where } n \text{ is a cardinal and } E_{0} \text{ is the minimal divisible extension of a product of groups } J_{p} \text{ for various primes } p. \text{ But since there exists a continuous monomorphism from } \hat{G}_{0} \text{ into } \hat{Q} \text{ there also exists a continuous monomorphism from } E \text{ into } \hat{Q} \text{ by Lemma 1. From this it follows that } n \leq c \text{ and that at most one group } J_{p} \text{ appears for each prime } p, \text{ since, as can be seen by examining the quotients of } Q, \text{ there}
do not exist continuous monomorphisms from $J^2_p$ into $\hat{Q}$. Hence $G_0$ is isomorphic to a closed subgroup of $Q^* \times \text{LP}(F_p)$. This means that $G$ is isomorphic to a closed subgroup of $R \times Q^* \times \text{LP}(F_p)$. The proof is completed by dualizing the preceding statement.

**Remark 1.** It is not hard to show that all solenoidal groups contain a dense copy of $Q$. It can also be shown, although we shall not do it here, that all $Q$-dense LCA groups contain a dense copy of $Q$, except for the proper quotients of $Q$ taken discrete.

We close with an application of our findings to the study of densely divisible groups, that is, groups containing a dense divisible subgroup. If an LCA group $G$ contains a dense divisible subgroup $D$, we seek conditions under which we may conclude that $G$ is itself divisible. One sufficient condition is that either $G$ or $G$ be compactly generated; this can be shown with the aid of the structure theorem for compactly generated LCA groups [3, 9.8]. Below we show that, if $D$ is not “too large”, the same conclusion can be drawn.

**Theorem 2.** Let $G$ be an LCA group containing a dense divisible subgroup of finite rank. Then $G$ is divisible.

**Proof.** We first recall that a divisible group of rank $n$ can be written as the direct sum of $n$ groups $D_1, \cdots, D_n$, where each $D_i$ is isomorphic either to $Q$ or $Z(p^n)$ for some prime $p$ [3, A.14]. Our proof proceeds by induction. If $G$ is an LCA group containing a dense divisible subgroup of rank 1, then $G$ is $Q$-dense (recall that $Z(p^n)$ is a quotient of $Q$), so that $G$ is divisible by Theorem 1. Next, assume that any LCA group containing a dense divisible subgroup of rank $n$ is divisible. Now let $G$ be an LCA group containing a dense divisible subgroup $D$ of rank $n+1$. We wish to show that $G$ is divisible. Write $D = D_1 \oplus \cdots \oplus D_n \oplus D_n+1$ as above, and set $D_* = D_1 \oplus \cdots \oplus D_n$. The closure $\hat{D}_*$ of $D_*$ in $G$ is, by the inductive assumption, a divisible subgroup of $G$. If $\hat{D}_* = G$ there is nothing to prove. Otherwise, it is sufficient to show that the quotient $G/\hat{D}_*$ is divisible, for then it follows immediately that $G$ is divisible. But if $\pi$ denotes the natural map from $G$ onto $G/\hat{D}_*$, it is easy to see that $\pi(D_{n+1})$ is dense in $G/\hat{D}_*$, so that $G/\hat{D}_*$ is $Q$-dense and hence divisible. This completes the proof.

**Remark 2.** Theorem 2 no longer remains valid if “finite” is replaced by “countable”. To see this, let $E$ be the minimal divisible extension of the group $J_p^p$, where $p$ is a prime; this group is described in [3, 25.32(c)] and [4, 2.2]. Since there exist continuous monomorphisms from $J_p^p$ into $\hat{Q}$, we can construct a continuous monomorphism from $J_p^p$ into $\hat{Q}^{\mathbb{N}_0}$. This monomorphism may be extended to a continuous monomorphism $f:E \to \hat{Q}^{\mathbb{N}_0}$ by Lemma 1. Hence the adjoint map $f^*:\hat{Q}^{\mathbb{N}_0} \to \hat{E}$ has dense image.
[3, 24.41]. Thus $\hat{E}$ contains a dense divisible subgroup of countable rank, but $\hat{E}$ is not divisible by [4, 2.7].

REFERENCES


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