INFINITE MATRICES AND INVARIANT MEANS

PAUL SCHAEFER

ABSTRACT. Let $\sigma$ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^n(n) \neq n$ for all positive integers $n$ and $p$, where $\sigma^p(n) = \sigma(\sigma^{p-1}(n))$, $p = 1, 2, \ldots$. A continuous linear functional $\varphi$ on the space of real bounded sequences is an invariant mean if $\varphi(x) \geq 0$ when the sequence $x = \{x_n\}$ has $x_n \geq 0$ for all $n$, $\varphi((1, 1, 1, \cdots)) = +1$, and $\varphi((x_{\sigma(n)})) = \varphi(x)$ for all bounded sequences $x$. Let $V_\sigma$ be the set of bounded sequences all of whose invariant means are equal. If $A = (a_{nk})$ is a real infinite matrix, then $A$ is said to be (1) $\sigma$-conservative if $Ax = \{\sum_{k} a_{nk}x_k\} \in V_\sigma$ for all convergent sequences $x$, (2) $\sigma$-regular if $Ax \in V_\sigma$ and $\varphi(Ax) = \lim x$ for all convergent sequences $x$ and all invariant means $\varphi$, and (3) $\sigma$-coercive if $Ax \in V_\sigma$ for all bounded sequences $x$. Necessary and sufficient conditions are obtained to characterize these classes of matrices.

1. Introduction. Let $\sigma$ be a mapping of the set of positive integers into itself. A continuous linear functional $\varphi$ on $m$, the space of real bounded sequences, is said to be an invariant mean or a $\sigma$-mean if and only if (1) $\varphi(x) \geq 0$ when the sequence $x = \{x_n\}$ has $x_n \geq 0$ for all $n$, (2) $\varphi(e) = 1$, where $e = \{1, 1, 1, \cdots\}$, and (3) $\varphi((x_{\sigma(n)})) = \varphi(x)$ for all $x \in m$. For certain kinds of mappings $\sigma$, every invariant mean $\varphi$ extends the limit functional on the space $c$ of real convergent sequences, in the sense that $\varphi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$ where $V_\sigma$ is the set of bounded sequences all of whose $\sigma$-means are equal.

When $\sigma(n) = n + 1$, the $\sigma$-means are the classical Banach limits on $m$ and $V_\sigma$ is the set of almost convergent sequences $[5]$. If $A = (a_{nk})$ is an infinite matrix with real entries such that $Ax = \{\sum_{k} a_{nk}x_k\}$ is an almost convergent sequence for every convergent sequence $x$, $A$ is said to be an almost conservative matrix $[4]$. When the common value of all Banach limits of $Ax$ is $\lim x$ for all $x \in c$, then the almost conservative matrix $A$ is said to be almost regular. J. P. King $[4]$ gave necessary and sufficient conditions that a matrix be almost conservative or almost regular. More
recently, Eizen and Laush [2] considered the class of almost coercive matrices, those for which $Ax$ is almost convergent for every bounded sequence $x$. In this paper we define analogous notions of $\sigma$-conservative, $\sigma$-regular, and $\sigma$-coercive matrices and obtain conditions which characterize them.

2. Preliminaries. We consider the spaces $c$ and $m$ as Banach spaces normed by $\|x\| = \sup\{|x_n|\}$. Let $c'$ and $m'$ denote the conjugate spaces of $c$ and $m$ respectively, normed in the usual way. It is well known that each $f \in c'$ has the representation

$$f(x) = (\lim x) \left[ f(e) - \sum_{k=1}^{\infty} f(e^k) \right] + \sum_{k=1}^{\infty} x_k f(e^k),$$

where $x = \{x_k\}$ and $e^k$ is the sequence having $+1$ in its $k$th entry and zeros elsewhere. Furthermore, $\|f\|$ is given by $|f(e) - \sum_{k=1}^{\infty} f(e^k)| + \sum_{k=1}^{\infty} |f(e^k)|$.

The set $\{e, e^2, e^3, \cdots\}$ is a Schauder basis for $c$, and every $x = \{x_k\} \in c$ can be written uniquely as $x = (\lim x)e + \sum_k (x_k - \lim x)e^k$ [8].

Throughout this paper we deal only with mappings $\sigma$ of the set of positive integers into itself which are one-to-one and are such that $\sigma^p(n) \neq n$ for all positive integers $n$ and $p$, where $\sigma^p(n)$ denotes the $p$th iterate of the mapping $\sigma$ at $n$. For such mappings, every $\sigma$-mean extends the limit functional on $c$ [6].

If $x = \{x_k\}$, set $Tx = \{x_{\sigma(n)}\}$. It can be shown that the set $V_\sigma$ described in the Introduction can be characterized as the set of all bounded sequences $x$ for which $\lim_{n} x = \frac{x + Tx + \cdots + T^{n}x}{(p+1)}$ exists in the space $m$ and has the form $L e$, $L$ being the common value of all $\sigma$-means at $x$ [6]. We write $L = \sigma$-lim $x$.

3. $\sigma$-conservative and $\sigma$-regular matrices. All matrices in this paper are real infinite matrices. For such matrices, the notions of being almost conservative and almost regular can be generalized as follows.

**Definition 1.** An infinite matrix $A$ is said to be $\sigma$-conservative if and only if $Ax = \{\sum_k a_{nk}x_k\} \in V_\sigma$ for all $x \in c$.

**Definition 2.** An infinite matrix $A$ is said to be $\sigma$-regular if and only if it is $\sigma$-conservative and $\sigma$-lim $Ax = \lim x$ for all $x \in c$.

**Theorem 1.** The matrix $A$ is $\sigma$-conservative if and only if

1. $\|A\| = \sup_k \{\sum |a_{nk}|\} < +\infty$,
2. $a_{nk} = \{a_{nk}\}_{n=1}^{\infty} \in V_\sigma$ for each $k$, and
3. $a = \{\sum a_{nk}\}_{n=1}^{\infty} \in V_\sigma$.

When $A$ is $\sigma$-conservative, the $\sigma$-limit of $Ax$ is $(\lim x)[u - \sum_k u_k] + \sum_k x_k u_k$ for every $x = \{x_k\} \in c$, where $u = \sigma$-lim $a$ and $u_k = \sigma$-lim $a_{(k)}$, $k = 1, 2, \cdots$.  

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
**Theorem 2.** The matrix $A$ is $\sigma$-regular if and only if

1. $\|A\| < +\infty$,
2. $a(\ell k) \in V_\sigma$ with $\sigma$-limit zero for each $k$, and
3. $a \in V_\sigma$ with $\sigma$-limit $+1$.

For typographical convenience we shall use the notation $a(n, k)$ to denote the element $a_{nk}$ of the matrix $A$ in the following proofs.

**Proof of Theorem 1.** Let us first suppose that conditions (1), (2) and (3) hold. Let $p$ be any nonnegative integer and let $x \in c$. We have

$$(Ax + TAx + \cdots + T^pAx)/(p + 1) = \left(\sum_k [a(n, k) + a(\sigma(n), k) + \cdots + a(\sigma^p(n), k)]x_k/(p + 1)\right)_{n=1}^\infty.$$ 

For every positive integer $n$, set

$$t_p(x) = \sum_{k=1}^\infty \sum_{j=0}^p a(\sigma^j(n), k)x_k/(p + 1).$$

Then we have

$$|t_p(x)| \leq \sum_{k=1}^\infty \sum_{j=0}^p |a(\sigma^j(n), k)| |x_k|/(p + 1) \leq \|x\|/(p + 1) \cdot \left[\sum_{j=0}^p \sum_{k=1}^\infty |a(\sigma^j(n), k)| \right] \leq \|A\| \cdot \|x\|.$$ 

Since $t_p(x)$ is obviously linear on $c$, it follows that $t_p(x) \in c'$ and that $\|t_p\| \leq \|A\|$. Now,

$$t_p(e) = \left[\sum_{k=1}^\infty \sum_{j=0}^p a(\sigma^j(n), k)\right]/(p + 1) = \left[\sum_{j=0}^p \sum_{k=1}^\infty a(\sigma^j(n), k)\right]/(p + 1),$$

so $\lim_p t_p(e)$ exists uniformly in $n$ and equals $u$, the $\sigma$-limit of $a$, since $a \in V_\sigma$. Similarly, $\lim_p t_p(e^k) = u_k$, the $\sigma$-limit of $a(\ell k)$ for each $k$, uniformly in $n$. Since $\{e, e^2, e^3, \cdots\}$ is a fundamental set in $c$, and $\sup_p |t_p(x)|$ is finite for each $x \in c$, it follows that $\lim_p t_p(x) = t_n(x)$ exists uniformly in $n \in c$ [1, p. 60]. Furthermore, $\|t_n\| \leq \liminf_p \|t_p\| \leq \|A\|$ for each $n$, and $t_n \in c'$. Thus,

$$t_n(x) = (\lim x) \left[\sum_k t_n(e^k)\right] + \sum_k x_k t_n(e^k)$$

$$= (\lim x) \left[u - \sum_k u_k\right] + \sum_k x_k u_k,$$

an expression independent of $n$. Denote this expression by $L(x)$.
In order to see that \( \lim_{n \to \infty} t_{pn}(x) = L(x) \) uniformly in \( n \), set \( F_{pn}(x) = t_{pn}(x) - L(x) \). Then \( F_{pn} \in c' \), \( \| F_{pn} \| \leq 2\| A \| \) for all \( p \) and \( n \), \( \lim_p F_{pn}(e) = 0 \) uniformly in \( n \), and \( \lim_p F_{pn}(e^k) = 0 \) uniformly in \( n \) for each \( k \). Let \( K \) be an arbitrary positive integer. Then

\[
x = (\lim x)e + \sum_{k=1}^{K} (x_k - \lim x)e^k + \sum_{k=K+1}^{\infty} (x_k - \lim x)e^k,
\]

so we have

\[
F_{pn}(x) = (\lim x)F_{pn}(e) + \sum_{k=1}^{K} (x_k - \lim x)F_{pn}(e^k) + F_{pn}\left( \sum_{k=K+1}^{\infty} (x_k - \lim x)e^k \right).
\]

Now,

\[
\left| F_{pn}\left( \sum_{k=K+1}^{\infty} (x_k - \lim x)e^k \right) \right| \leq 2\| A \| \cdot \sup_{k \geq K+1} \{|x_k - \lim x|\}
\]

for all \( p \) and \( n \). By first choosing a fixed \( K \) large enough, it is easy to see that each of the three displayed terms for \( F_{pn}(x) \) can be made to be uniformly small in absolute value for all sufficiently large \( p \), so \( \lim_p F_{pn}(x) = 0 \) uniformly in \( n \). This shows that

\[
\lim_p (Ax + TAx + \cdots + T^pAx)/(p + 1) = L(x)e,
\]

so that \( Ax \in V_\sigma \) and the matrix \( A \) is \( \sigma \)-conservative.

Conversely, suppose that \( A \) is \( \sigma \)-conservative. If \( x \) is any null sequence, then \( Ax \in V_\sigma \subset m \). It follows from the proof of [3, Theorem 1, pp. 45 and 46] that \( \| A \| < +\infty \). Furthermore, since \( Ae = a \) and \( A^k = a^k \), the other two conditions are necessary for \( \sigma \)-conservative matrices.

**Proof of Theorem 2.** If a matrix \( A \) satisfies the three conditions of the theorem, then it is a \( \sigma \)-conservative matrix. For \( x \in c \), the \( \sigma \)-limit of \( Ax \) is \( L(x) \), which reduces to \( \lim x \), since \( u = 1 \) and \( u_k = 0 \) for each \( k \). Hence, \( A \) is a \( \sigma \)-regular matrix. Conversely, if \( A \) is \( \sigma \)-regular, then \( \sigma \)-lim \( Ax \) = +1 = \( \sigma \)-lim \( a \), \( \sigma \)-lim \( A^k = \sigma \)-lim \( a^k \), and \( \| A \| \) is finite, as in the proof of Theorem 1.

**4. \( \sigma \)-coercive matrices.**

**Definition 3.** A matrix \( A \) is \( \sigma \)-coercive if and only if \( Ax \in V_\sigma \) for all \( x \in m \).

**Theorem 3.** The matrix \( A \) is \( \sigma \)-coercive if and only if

1. \( \| A \| \) is finite,
2. \( a_{ij} \in V_\sigma \) for each \( k \), and
3. \( \lim_p \sum_{k=1}^{\infty} | \sum_{j=0}^{p} (a_\sigma (n) - u_k)/(p+1) = 0 \) uniformly in \( n \), where
In this case, the $\sigma$-limit of $Ax$ is $\sum_k u_k x_k$ for every $x = \{x_k\} \in m$.

**Proof.** Let us first assume that the matrix $A$ satisfies conditions (1), (2) and (3). For any positive integer $K$,

$$\sum_{k=1}^{K} |u_k| = \sum_{k=1}^{K} \lim_{p \to \infty} \left| \sum_{j=0}^{p} a(\sigma'(n), k) \right| / (p + 1)$$

$$= \lim_{p \to \infty} \sum_{k=1}^{K} \left| \sum_{j=0}^{p} a(\sigma'(n), k) \right| / (p + 1)$$

$$\leq \lim \sup_{p} \sum_{j=0}^{\infty} \sum_{k=1}^{K} |a(\sigma'(n), k)| / (p + 1) \leq \|A\|.$$  

This shows that $\sum_{k=1}^{\infty} |u_k|$ converges, and that $\sum_k u_k x_k$ is defined for every bounded sequence $x = \{x_k\}$.

Let $x$ be an arbitrary bounded sequence. For every positive integer $p$,

$$(Ax + TAx + \cdots + T^nAx) / (p + 1) - \left( \sum_k u_k x_k \right) e$$

$$= \left\{ \sum_{k=1}^{\infty} \left( \sum_{j=0}^{p} \left[ a(\sigma'(n), k) - u_k \right] / (p + 1) \right) x_k \right\},$$

so

$$\left\| (Ax + TAx + \cdots + T^nAx) / (p + 1) - \left( \sum_k u_k x_k \right) e \right\|$$

$$= \sup_n \left\| \sum_{k=1}^{\infty} \left( \sum_{j=0}^{p} \left[ a(\sigma'(n), k) - u_k \right] / (p + 1) \right) x_k \right\|$$

$$\leq \|x\| \cdot \sup_n \left\| \sum_{k=1}^{\infty} \sum_{j=0}^{p} \left[ a(\sigma'(n), k) - u_k \right] / (p + 1) \right\|.$$

Let $p \to \infty$. By the uniformity of the limits in condition (3), it follows that $(Ax + TAx + \cdots + T^nAx) / (p + 1) \to (\sum_k u_k x_k) e$, and that $Ax \in V_\sigma$ with $\sigma$-limit $\sum_k u_k x_k$.

Next, suppose that $A$ is a $\sigma$-coercive matrix. Then, since $A$ is $\sigma$-conservative, we have conditions (1) and (2) from Theorem 1. In order to see that (3) holds, we proceed as in [2], by first showing that the limit in question is zero for each $n$, and secondly showing that the limit is uniform in $n$.

Thus, suppose that for some $n$, we have

$$\lim \sup_{p} \sum_{k=1}^{\infty} \left| \sum_{j=0}^{p} \left[ a(\sigma'(n), k) - u_k \right] / (p + 1) \right| = N > 0.$$
Since $\|A\|$ is finite, $N$ is finite also. We observe that since $\sum |u_k| < +\infty$, the matrix $B = (b_{nk})$, where $b_{nk} = a_{nk} - u_k$, is also a $\sigma$-coercive matrix. If one sets $F_{kp} = \sum_{j=0}^{\infty} [a(\sigma^j(n), k) - u_k] \left/ (p+1) \right.$, and $E_{kp} = F_{kp} \cdot \mathbf{1}$, one can follow the construction in the proof of Theorem 2.1 in [2] to obtain a bounded sequence whose transform by the matrix $B$ is not in $V_\sigma$. This contradiction shows that the limit in (3) is zero for every $n$.

To show that this convergence is uniform in $n$, we invoke the following lemma, which is proved in [7].

**Lemma.** Let $\{H(n)\}$ be a countable family of matrices $H(n) = (h_{pk}(n))$ such that $\|H(n)\| \leq M < +\infty$ for all $n$ and $\lim_{n} h_{pk}(n) = 0$ for each $k$, uniformly in $n$. Then $\lim_{n} \sum_{k} h_{pk}(n)x_k = 0$ uniformly in $n$ for all $x \in m$ if and only if $\lim_{n} \sum_{k} |h_{pk}(n)| = 0$ uniformly in $n$.

We let $h_{pk}(n) = \sum_{j=0}^{\infty} [a(\sigma^j(n), k) - u_k] \left/ (p+1) \right.$ and let $H(n)$ be the matrix $(h_{pk}(n))$. It is easy to see that $\|H(n)\| \leq 2 \cdot \|A\|$ for every $n$, and that $\lim_{n} h_{pk}(n) = 0$ for each $k$, uniformly in $n$ by condition (2). For any $x \in m$, $\lim_{n} \sum_{k} h_{pk}(n)x_k = \sigma\lim A x - \sum_{k} u_k x_k$, and the limit exists uniformly in $n$ since $A x \in V_\sigma$. Moreover this limit is zero since

$$\left| \sum_{k} h_{pk}(n)x_k \right| \leq \|x\| \cdot \sum_{k} \left| \sum_{j=0}^{\infty} [a(\sigma^j(n), k) - u_k] \right| \left/ (p+1) \right.$$ 

Thus, $\lim_{n} \sum_{k} |h_{pk}(n)| = 0$ uniformly in $n$, and the matrix $A$ satisfies condition (3).

**Theorem 4.** The classes of $\sigma$-regular and $\sigma$-coercive matrices are disjoint.

**Proof.** If $A$ were a $\sigma$-regular and a $\sigma$-conservative matrix, then $\sigma\lim a_{(k)} = 0 = u_k$ for every $k$. The conditions $\sigma\lim a = +1$ and

$$\lim_{n} \sum_{k} \left| \sum_{j=0}^{\infty} a(\sigma^j(n), k) \right| = 0$$

are incompatible, since

$$\left| \frac{1}{p+1} \sum_{j=0}^{\infty} \sum_{k} a(\sigma^j(n), k) \right| = \left| \sum_{k} \frac{1}{p+1} \sum_{j=0}^{\infty} a(\sigma^j(n), k) \right| \leq \sum_{k} \frac{1}{p+1} \left| \sum_{j=0}^{\infty} a(\sigma^j(n), k) \right|.$$
References


Department of Mathematics, State University College of New York, Geneseo, New York 14454