

ON THE MÜNTZ-SZÁSZ THEOREM FOR $C[0, 1]$

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ABSTRACT. The functions $1, t^{\lambda_1}, t^{\lambda_2}, \dots$ with complex λ 's are shown to be incomplete in $C[0, 1]$ under conditions weaker than those proven by Szász, and a special construction due to P. D. Lax where the functions are complete is given.

In 1916 Szász proved the following classical result:

THEOREM 1. *Suppose $\operatorname{Re} \lambda_j > 0, j = 1, 2, \dots$, and, for the sake of simplicity, the λ 's are distinct. Then the functions $1, t^{\lambda_1}, t^{\lambda_2}, \dots$ are complete in $C[0, 1]$ if*

$$(1) \quad \sum \frac{\operatorname{Re} \lambda_j}{1 + |\lambda_j|^2} = \infty$$

and incomplete if

$$(2) \quad \sum \frac{\operatorname{Re} \lambda_j + 1}{1 + |\lambda_j|^2} < \infty.$$

When $\liminf_{j \rightarrow \infty} \operatorname{Re} \lambda_j > 0$, (1) is the negation of (2) and constitutes a necessary and sufficient condition for completeness. However, under circumstances where neither (1) nor (2) is satisfied, for example if $\lambda_j = 1/j + i\sqrt{j}$ or the λ 's are bounded and $\sum \operatorname{Re} \lambda_j < \infty$, the completeness question is not answered by Szász's theorem.

For the L^2 case, it should be noted that there is no such unresolved margin. This is because the L^2 distance between t^λ and the linear span of $\{t^{\lambda_j}\}_{j=1}^\infty$ can be explicitly expressed in terms of a Blaschke product which converges to a nontrivial function if and only if

$$(3) \quad \sum \frac{\operatorname{Re} \lambda_j + \frac{1}{2}}{1 + |\lambda_j|^2} < \infty.$$

The purpose of this paper is to reduce the gap in Theorem 1.

THEOREM 2. *Suppose $\operatorname{Re} \lambda_j > 0, j = 1, 2, \dots, \lambda_j \rightarrow \infty$ as $j \rightarrow \infty$ and the λ 's are distinct. If, for some $\alpha < 1$,*

$$(4) \quad \sum \frac{\operatorname{Re} \lambda_j + \exp(-|\lambda_j|^\alpha)}{1 + |\lambda_j|^2} < \infty,$$

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then the linear span of $1, t^{\lambda_1}, t^{\lambda_2}, \dots$ is not dense in $C[0, 1]$. In fact, t^λ cannot be uniformly approximated by elements of the span unless λ equals one of the λ_j 's or 0.

PROOF. We shall construct a bounded linear functional l on $C[0, 1]$, see (14), such that $l(t^\lambda) = 0$ only when $\lambda = 0, \lambda_1, \lambda_2, \dots$. First, a few preliminary constructions. Set

$$(5) \quad e^{-s} = t, \quad F(s) = f(e^{-s})$$

for $f \in C[0, 1]$. Then $F(s) \in C[0, \infty]$ and has a Laplace transform

$$(6) \quad \hat{F}(w) = \int_0^\infty e^{ws} F(s) ds$$

for $\operatorname{Re} w < 0$. Clearly \hat{F} satisfies

$$(7) \quad |\hat{F}(w)| \leq \frac{\|F\|_\infty}{-\operatorname{Re} w}.$$

LEMMA. Choose some $\beta, \alpha < \beta < 1$, and, via the Poisson integral, define an analytic function ψ in the right half-plane such that

$$(8) \quad \operatorname{Re} \psi(iy) = \exp(-|y|^\beta).$$

Then ψ has the following properties:

- (a) ψ is bounded.
- (b) The mapping $z \rightarrow w$ defined by

$$(9) \quad w = z - \psi(z)$$

maps the right half-plane 1-1 onto a region containing $\operatorname{Re} w \geq 0$.

- (c) Denote by γ_j the point whose image under this map is λ_j :

$$(10) \quad \lambda_j = \gamma_j - \psi(\gamma_j).$$

Then

$$(11) \quad \operatorname{Re} \gamma_j = O(\operatorname{Re} \lambda_j + \exp(-|\lambda_j|^\alpha)).$$

Proof later.

It follows from (4) and (11) that the Blaschke product

$$(12) \quad B(z) = \prod_{j=0}^{\infty} \frac{\gamma_j - z}{\bar{\gamma}_j + z} \frac{\bar{\gamma}_j}{\gamma_j} = \prod_{j=0}^{\infty} \left(1 - \frac{2z \operatorname{Re} \gamma_j}{|\gamma_j|^2 + \gamma_j z} \right)$$

converges. From (7) and (8), we have

$$(13) \quad |\hat{F}(iy - \psi(iy))| \leq \exp(|y|^\beta) \|F\|_\infty.$$

Choosing a δ , $\alpha < \beta < \delta < 1$, we define the linear functional $l(f)$ as follows:

$$(14) \quad l(f) = \int_{-i\infty}^{i\infty} \hat{F}(z - \psi(z)) \exp(-z^\delta) B(z) dz.$$

Since $|B(iy)|=1$ and $|\exp(|y|^\beta)\exp(-(iy)^\delta)|$ is integrable, it follows from (13) that l is bounded. Now set $f=t^\lambda$; then $F(s)=e^{-\lambda s}$, $\hat{F}(w)=1/(\lambda-w)$, and $\hat{F}(z-\psi(z))=-1/(z-\psi(z)-\lambda)$ is meromorphic with one pole, γ , in the right half-plane. In this case, the contour in (14) can be shifted to the right, with the total contribution resulting from the residue at $z=\gamma$ since the integrand vanishes in a dominated way at infinity:

$$(15) \quad l(t^\lambda) = 2\pi i \frac{\exp(-\gamma^\delta)B(\gamma)}{1 - \psi(\gamma)}.$$

Therefore $l(t^\lambda)$ vanishes if and only if B vanishes at γ ; this happens only when $\gamma=\gamma_0, \gamma_1, \dots$, i.e., when $\lambda=0, \lambda_1, \lambda_2, \dots$. This completes the proof except for the lemma.

Setting

$$\psi(x + iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-|t|^\beta}}{x + i(y - t)} dt$$

gives property (8) and the boundedness of $\text{Re } \psi$. But

$$\begin{aligned} |\text{Im } \psi(x + iy)| &= \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-|t|^\beta}(t - y)}{x^2 + (y - t)^2} dt \right| \\ &= \left| \frac{1}{\pi} \int_0^{\infty} \frac{t(e^{-|t+y|^\beta} - e^{-|t-y|^\beta})}{x^2 + t^2} dt \right| \\ &< \sup_y \int_0^1 \left| \frac{e^{-|t+y|^\beta} - e^{-|t-y|^\beta}}{t} \right| dt + \int_{-\infty}^{\infty} e^{-|t|^\beta} dt \\ &< 2\beta \sup_y \int_0^1 \frac{e^{-|t-y|^\beta}}{|t - y|^{1-\beta}} dt + \int_{-\infty}^{\infty} e^{-|t|^\beta} dt < \infty. \end{aligned}$$

So $|\psi(z)| < k$ for some k , which is property (a). Applying the argument principle to $z - \psi(z)$ in the right half-plane gives (b). Now

$$\begin{aligned} \text{Re } \psi(z) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x e^{-|t|^\beta}}{x^2 + (y - t)^2} dt \\ &\leq \frac{1}{\pi} \int_{-|y/2|}^{|y/2|} \frac{e^{-|t|^\beta} x}{x^2 + (y - t)^2} dt + \frac{e^{-|y/2|^\beta}}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y - t)^2} dt \\ &< \frac{4x|y|}{\pi(x^2 + y^2)} + e^{-|y/2|^\beta}, \end{aligned}$$

(16) $\text{Re } \psi(z) < \text{Re}(z/2) + \exp(-|\text{Im}(z/2)|^\beta), \quad |z| \text{ large.}$

But $\gamma_j = \lambda_j + \psi(\gamma_j)$, and hence $\text{Re } \gamma_j = \text{Re } \lambda_j + \text{Re } \psi(\gamma_j)$. Applying (16) yields

$$(17) \quad \text{Re } \gamma_j = O(\text{Re } \lambda_j + \exp(-|\text{Im}(\gamma_j/2)|^\beta)).$$

Since $|\lambda_j - \gamma_j| < k$, it follows that

$$(18) \quad \exp(-|\text{Im}(\gamma_j/2)|^\beta) = O(\text{Re } \lambda_j + \exp(-|\gamma_j|^\alpha)), \quad \alpha < \beta.$$

Combining (17) and (18) gives (11), the final part of the lemma.

COROLLARY. *Suppose $\text{Re } \lambda_j > 0, j = 1, 2, \dots, \lambda_j \rightarrow i\gamma$ as $j \rightarrow \infty$ and the λ_j 's are distinct. If, for some $\alpha < 1$,*

$$(19) \quad \sum \text{Re } \lambda_j + \exp\left(-\left|\frac{1}{\lambda_j - i\gamma}\right|^\alpha\right) < \infty,$$

then $1, t^{\lambda_1}, t^{\lambda_2}, \dots$ is not complete in $C[0, 1]$.

This follows from setting

$$(20) \quad l(f) = \int_{-i\infty}^{i\infty} \hat{F}\left(i\gamma + \frac{1}{z - \psi(z)}\right) e^{-z^\delta} B(z) dz.$$

COROLLARY. *Suppose $\lambda_{j,k} \rightarrow i\gamma_k$ and $\lambda_{j,0} \rightarrow \infty$ as $j \rightarrow \infty, k = 1, 2, \dots, l$, the λ 's are distinct and with positive real parts. If, for some $\alpha < 1$,*

$$(21) \quad \sum_{j=1}^{\infty} \text{Re} \left[\frac{\lambda_{j,0} + \exp(-|\lambda_{j,0}|^\alpha)}{1 + |\lambda_{j,0}|^2} + \sum_{k=1}^l \lambda_{j,k} + \exp\left(-\left|\frac{1}{\lambda_{j,k} - i\gamma_k}\right|^\alpha\right) \right] < \infty,$$

then the functions $1, t^{\lambda_{j,k}}, j = 1, 2, \dots, k = 0, 1, \dots, l$, are not complete in $C[0, 1]$.

The result is a direct consequence of setting

$$(22) \quad \hat{F}(w_0, w_1, \dots, w_l) = \int_0^\infty \int_0^\infty \dots \int_0^\infty \exp(w_0 s_0 + w_1 s_1 + \dots + w_l s_l) \times F(s_0 + s_1 + \dots + s_l) ds_0 ds_1 \dots ds_l,$$

$$(23) \quad l(f) = \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} \hat{F}\left(z_0 - \psi(z_0), i\gamma_1 + \frac{1}{z_1 - \psi(z_1)}, \dots, i\gamma_l + \frac{1}{z_l - \psi(z_l)}\right) \times \prod_{k=0}^l e^{-z_k^\delta} B_k(z_k) dz_k.$$

COROLLARY. *Suppose the λ_j 's are positive and distinct for $j = 1, 2, \dots$. Then $1, t^{\lambda_1}, t^{\lambda_2}, \dots$ are complete in $C[0, 1]$ if and only if*

$$(24) \quad \sum [\lambda_j / (1 + \lambda_j^2)] = \infty.$$

This generalization of Müntz's theorem is stated, though not proven, in [2, p. 29]. Its proof may be simplified by letting $\psi \equiv 1$ in the preceding calculations.

For complex λ 's, however, Lax has shown by example that (1) is not necessary for completeness.

THEOREM 3 (LAX). *Let l_j be any sequence of positive numbers going to infinity, and α_j any positive sequence tending to zero. Define $\lambda_{j,k}$ by*

$$(25) \quad \lambda_{j,k} = \alpha_j + i2\pi k/l_j, \quad j = 1, 2, \dots, k = 0, \pm 1, \pm 2, \dots$$

Then $\{1, t^{\lambda_{j,k}}\}_{j=1}^{\infty} \{k=-\infty}^{\infty}$ is complete in $C[0, 1]$.

If we choose α_j and l_j so that

$$(26) \quad \sum \alpha_j l_j < \infty,$$

then $\sum_{j,k} [\text{Re } \lambda_{j,k} / (1 + |\lambda_{j,k}|^2)]$ converges since

$$\sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} \frac{\text{Re } \lambda_{j,k}}{1 + |\lambda_{j,k}|^2} < \sum_{j=1}^{\infty} \alpha_j \left(1 + 2 \int_0^{\infty} \frac{dx}{1 + (x/l_j)^2} \right) = O(\sum \alpha_j l_j).$$

Unfortunately, this example is somewhat pathological since the accumulation points of $\lambda_{j,k}$ are precisely the imaginary axis.

PROOF OF THEOREM 3. Let X be the closure in $C_0[0, \infty]$ of the linear space spanned by $\{e^{-\lambda_{j,k}s}\}_{j=1}^{\infty} \{k=-\infty}^{\infty}$, where

$$C_0[0, \infty] = \left\{ F \in C[0, \infty] : \lim_{s \rightarrow \infty} F(s) = 0 \right\}.$$

To prove that $X = C_0[0, \infty]$, an equivalent formulation of the theorem, it suffices to show that for any $F \in C_0[0, \infty]$, there is a $G \in X$ such that

$$(27) \quad \|F - G\|_{\infty} \leq \frac{2}{3} \|F\|_{\infty}.$$

This is because successive approximations give a sequence in X converging to F .

Let F be an element of $C_0[0, \infty]$. Since $\lim_{s \rightarrow \infty} F(s) = 0$, we may choose J so large that

$$(28) \quad |F(s)| \leq \frac{1}{3} \|F\|_{\infty} \quad \text{for all } s > l_J - 1.$$

For convenience, denote l_J by l , and α_J by α . Define $H(s)$ on $[0, l]$ by

$$(29) \quad \begin{aligned} H(s) &= e^{\alpha s} F(s), & 0 \leq s < l - 1, \\ &= e^{\alpha s} F(s)(l - s) + F(0)(s - l + 1), & l - 1 \leq s \leq l. \end{aligned}$$

H is continuous, and $H(0) = H(l)$. Define L to be the periodic extension of H . Then L can be uniformly approximated by linear combinations of

$\exp(i2\pi ks/l)$, $k=0, \pm 1, \pm 2, \dots$. Therefore $G(s)=\frac{1}{3}e^{-as}L(s)$ can be uniformly approximated by linear combinations of $\exp(-\lambda_{J,k}s)$, $k=0, \pm 1, \pm 2, \dots$, and hence belongs to X . Clearly

$$(30) \quad G(s) = \frac{1}{3}F(s) \quad \text{for } s \in [0, l-1].$$

For $s \in [l-1, l]$,

$$(31) \quad \begin{aligned} |G(s)| &\leq \frac{1}{3}(|F(s)|(l-s) + |F(0)|(s-l+1)) \\ &\leq \frac{1}{3}(\|F\|_{\infty}(l-s) + \|F\|_{\infty}(s-l+1)) = \frac{1}{3}\|F\|_{\infty}. \end{aligned}$$

Since $|G(s)| \leq |G(s-l)|$ for $s > l$, it follows that

$$(32) \quad \|G\|_{\infty} = \frac{1}{3}\|F\|_{\infty}.$$

From (30) we have

$$(33) \quad |F(s) - G(s)| \leq \frac{2}{3}\|F\|_{\infty}, \quad s \in [0, l-1].$$

Combining (32) and (28) gives

$$(34) \quad |F(s) - G(s)| \leq |F(s)| + |G(s)| \leq \frac{2}{3}\|F\|_{\infty}, \quad s > l-1.$$

Combining (33) and (34) we see that $\|F-G\|_{\infty} \leq \frac{2}{3}\|F\|_{\infty}$, as asserted in (27).

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