ON THE MEAN VALUE OF A WEAKLY ALMOST PERIODIC FUNCTION

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Abstract. Let $M$ denote the invariant mean on the space $W(G)$ of weakly almost periodic functions on a LCA group $G$. The purpose of this note is to show that, for each $\phi \in W(G)$,

$$M(\phi) = \lim_{V \to \{1\}} \int \hat{f}_V(x) \phi(x) \, dx$$

where $\{V\}$ is the system of compact neighborhoods of 1 in the character group $\Gamma$, and, for each $V$, $f_V$ is a continuous positive definite function supported in $V$ and satisfying $f_V(1) = 1$. This answers affirmatively a question recently raised by R. Burckel.

In his recent monograph [1] on w.a.p. functions, R. Burckel shows that functions $f_V$, as above, can be found so that (1) holds for every w.a.p. function $\phi$ which is the uniform limit of Fourier-Stieltjes transforms, and he asks [1, p. 81, Question 6] whether the same formula is valid for all $\phi \in W(G)$. We will show that it is.

Our proof depends on a simple fact which, while undoubtedly well known, does not seem to be readily available in the literature in the form that we need. We include it here for the sake of completeness. For $x \in G$, we use $\delta_x * f$ to denote the $x$-translate of a function $f$ on $G$.

Lemma. Let $\{f_\alpha\}$ be a net of nonnegative functions in $L_1(G)$, normalized in the sense that $\int_G f_\alpha(x) \, dx = 1$ for all $\alpha$, and satisfying the condition

(i) $\lim_\alpha \|\delta_x * f_\alpha - f_\alpha\|_1 = 0$ for all $x \in G$.

Then, for all $\phi \in W(G)$, we have

(ii) $M(\phi) = \lim_\alpha \int_G f_\alpha(x) \phi(x) \, dx$.

Proof. For each $\alpha$, regard $f_\alpha$ as an element of the unit ball, $\Sigma$, of the dual of $W(G)$. Since $\Sigma$ is weak*-compact, the net $\{f_\alpha\}$ must have at least one weak* cluster point in $\Sigma$. Furthermore, using (i), it is easy to verify that any such cluster point is an invariant mean on $W(G)$, hence must coincide with $M$. Thus $\{f_\alpha\}$ converges weak* to $M$, i.e. (ii) holds for all $\phi \in W(G)$.

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Now let \( \{V\} \) denote the system of compact neighborhoods of 1 in the character group \( \Gamma \) of \( G \). For each \( V \), let \( W \) be a compact neighborhood of 1 such that \( WW^{-1} \subseteq V \), and let \( f_V = (g_V) \ast (g_V)^* \) where \( g_V \) is a continuous function vanishing outside \( W \), with \( \|g_V\|_2 = 1 \).

Then \( f_V \) is positive definite, \( f_V \) vanishes outside \( V \), and \( f_V(1) = \|g_V\|_2 = 1 \). Thus \( f_V \) is nonnegative, and \( \int_G f_V(x) \, dx = f_V(1) = 1 \) by the Fourier Inversion Theorem. We may (and do) require, in addition, that the \( g_V \)'s are themselves positive definite. Thus \( \hat{g}_V \) is nonnegative, and \( f_V = (\hat{g}_V)^2 \).

**Theorem.** \( M(\phi) = \lim_{V \to \{V\}} \int_G f_V(x) \phi(x) \, dx \) for all \( \phi \in \mathcal{W}(G) \).

**Proof.** We have only to show that the net \( \{f_V\} \) satisfies condition (i) of the Lemma.

Let \( x \) be a fixed element of \( G \), and let \( \epsilon > 0 \). Choose \( V_0 \in \{V\} \) such that \( |\gamma(x) - 1| < \epsilon/2 \) for all \( \gamma \in V_0 \). Then if \( V \subseteq V_0 \), we have

\[
\|\delta_x \ast (f_V) - f_V\|_1 = \int_G |\hat{g}_V(x^{-1}y) - \hat{g}_V(y)| \, dy
\]

\[
\leq \|\delta_x \ast \hat{g}_V + \hat{g}_V\|_2 \|\delta_x \ast \hat{g}_V - \hat{g}_V\|_2
\]

\[
= \|xg_V + g_V\|_2 \|xg_V - g_V\|_2
\]

\[
\leq 2 \left( \int_{V_0} \left| \gamma(x) - 1 \right|^2 |g_V(\gamma)|^2 \, d\gamma \right)^{1/2}
\]

\[
\leq 2 \left( \frac{\epsilon^2}{4} \int_{V_0} |g_V(\gamma)|^2 \, d\gamma \right)^{1/2} = \epsilon \|g_V\|_2 = \epsilon.
\]

Thus \( \lim_{V \to \{V\}} \|\delta_x \ast f_V - f_V\|_1 = 0 \), and our proof is complete.

**References**


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