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ON THE MEAN VALUE OF A WEAKLY ALMOST PERIODIC FUNCTION

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ABSTRACT. Let M denote the invariant mean on the space $W(G)$ of weakly almost periodic functions on a LCA group G. The purpose of this note is to show that, for each $\phi \in W(G)$,

$$M(\phi) = \lim_{V \to \{1\}} \int_G \hat{f}_V(x)\phi(x)\,dx \tag{1}$$

where $\{V\}$ is the system of compact neighborhoods of 1 in the character group Γ, and, for each V, f_V is a continuous positive definite function supported in V and satisfying $f_V(1)=1$. This answers affirmatively a question recently raised by R. Burckel.

In his recent monograph [**1**] on w.a.p. functions, R. Burckel shows that functions f_V, as above, can be found so that (1) holds for every w.a.p. function ϕ which is the uniform limit of Fourier-Stieltjes transforms, and he asks [**1**, p. 81, Question 6] whether the same formula is valid for all $\phi \in W(G)$. We will show that it is.

Our proof depends on a simple fact which, while undoubtedly well known, does not seem to be readily available in the literature in the form that we need. We include it here for the sake of completeness. For $x \in G$, we use $\delta_x * f$ to denote the x-translate of a function f on G.

LEMMA. *Let $\{f_\alpha\}$ be a net of nonnegative functions in $L_1(G)$, normalized in the sense that $\int_G f_\alpha(x)\,dx=1$ for all α, and satisfying the condition*

(i) $\lim_\alpha \|\delta_x * f_\alpha - f_\alpha\|_1 = 0$ *for all* $x \in G$.

Then, for all $\phi \in W(G)$, we have

(ii) $M(\phi)=\lim_\alpha \int_G f_\alpha(x)\phi(x)\,dx$.

PROOF. For each α, regard f_α as an element of the unit ball, Σ, of the dual of $W(G)$. Since Σ is weak*-compact, the net $\{f_\alpha\}$ must have at least one weak* cluster point in Σ. Furthermore, using (i), it is easy to verify that any such cluster point is an invariant mean on $W(G)$, hence must coincide with M. Thus $\{f_\alpha\}$ converges weak* to M, i.e. (ii) holds for all $\phi \in W(G)$.

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Now let $\{V\}$ denote the system of compact neighborhoods of 1 in the character group Γ of G. For each V, let W be a compact neighborhood of 1 such that $WW^{-1}\subset V$, and let $f_V=(g_V)*(g_V)^*$ where g_V is a continuous function vanishing outside W, with $\|g_V\|_2=1$.

Then f_V is positive definite, f_V vanishes outside V, and $f_V(1)=\|g_V\|_2^2=1$. Thus $\hat{f}_V$ is nonnegative, and $\int_G \hat{f}_V(x)\,dx=f_V(1)=1$ by the Fourier Inversion Theorem. We may (and do) require, in addition, that the g_V's are themselves positive definite. Thus $\hat{g}_V$ is nonnegative, and $\hat{f}_V=(\hat{g}_V)^2$.

THEOREM. *$M(\phi)=\lim_{V\to\{1\}}\int_G \hat{f}_V(x)\phi(x)\,dx$ for all $\phi\in W(G)$.*

PROOF. We have only to show that the net $\{\hat{f}_V\}$ satisfies condition (i) of the Lemma.

Let x be a fixed element of G, and let $\varepsilon>0$. Choose $V_0\in\{V\}$ such that $|\gamma(x)-1|<\varepsilon/2$ for all $\gamma\in V_0$. Then if $V\subset V_0$, we have

$$\begin{aligned}
\|\delta_x*(\hat{f}_V)-\hat{f}_V\|_1 &= \int_G |\hat{g}_V^2(x^{-1}y)-\hat{g}_V^2(y)|\,dy\\
&= \int_G (\hat{g}_V(x^{-1}y)+\hat{g}_V(y))\,|\hat{g}_V(x^{-1}y)-\hat{g}_V(y)|\,dy\\
&\leqq \|\delta_x*\hat{g}_V+\hat{g}_V\|_2\,\|\delta_x*\hat{g}_V-\hat{g}_V\|_2\\
&= \|xg_V+g_V\|_2\,\|xg_V-g_V\|_2\\
&\leqq 2\left(\int_{V_0}|\gamma(x)-1|^2\,|g_V(\gamma)|^2\,d\gamma\right)^{1/2}\\
&\leqq 2\left(\frac{\varepsilon^2}{4}\int_{V_0}|g_V(\gamma)|^2\,d\gamma\right)^{1/2}=\varepsilon\,\|g_V\|_2=\varepsilon.
\end{aligned}$$

Thus $\lim_{V\to\{1\}}\|\delta_x*\hat{f}_V-\hat{f}_V\|_1=0$, and our proof is complete.

REFERENCES

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