THE OPERATOR EQUATION $THT = K$

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Abstract. Let $H$ and $K$ be bounded positive operators on a Hilbert space, and assume that $H$ is nonsingular. Then (i) there is at most one bounded positive operator $T$ such that $THT = K$; (ii) a necessary and sufficient condition for the existence of such $T$ is that $(H^{1/2}KH^{1/2})^{1/2} \leq aH$ for some $a > 0$, and then $\|T\| \leq a$; (iii) this condition is satisfied if $H$ is invertible or more generally if $K \leq a^2H$ for some $a > 0$; (iv) an exact formula for $T$ is given when $H$ is invertible.

If $H$ is a selfadjoint positive nuclear operator on a Hilbert space $\mathcal{H}$, then the map $\phi : A \rightarrow \text{Tr}(AH)$ is a normal positive functional on the von Neumann algebra $B(\mathcal{H})$. If $0 \leq K \leq H$ then the functional $\psi : A \rightarrow \text{Tr}(AK)$ is majorized by $\phi$. By S. Sakai's noncommutative Radon-Nikodym theorem [3] there is therefore a positive operator $T$ with $\|T\| \leq 1$ such that $\psi(A) = \phi(TAT)$ for all $A$ in $B(\mathcal{H})$. Moreover, by [4, Lemma 15.4] the operator $T$ is uniquely determined. Since the correspondence between normal positive functionals and positive nuclear operators is bijective this implies that $THT = K$. The purpose of this paper is to give a necessary and sufficient condition for the existence of a positive solution to the operator equation $THT = K$, with arbitrary $H$ and $K$ in $B(\mathcal{H})$. Applications of the result to noncommutative integration theory can be found in [2].

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Theorem. Let $H$ and $K$ be selfadjoint positive operators in $B(\mathcal{H})$, and assume that $H$ is nonsingular. There is then at most one positive operator $T$ in $B(\mathcal{H})$ such that $THT = K$. A necessary and sufficient condition for the existence of such $T$ is that $(H^{1/2}KH^{1/2})^{1/2} \leq aH$ for some $a > 0$; and then $\|T\| \leq a$. This condition will be satisfied if $H$ is invertible or, more generally, if $K \leq a^2H$ for some $a > 0$. 

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Proof. Suppose that $S$ and $T$ are positive operators in $B(\mathcal{S})$ such that $SHS = THT$. Put $A = H^{1/2}S$ and $B = H^{1/2}T$. Then $A*A = B*B$ and from the polar decomposition $A = UB$, where $U$ is a partial isometry such that $U*U$ is the range projection of $B$. Thus

$$H^{1/2}SH^{1/2} = A H^{1/2} = U B H^{1/2} = U H^{1/2} T H^{1/2}.$$ 

But $H^{1/2}SH^{1/2}$ and $H^{1/2}T H^{1/2}$ are both positive and since the polar decomposition (of $H^{1/2}SH^{1/2}$) is unique this implies that $U$ is the range projection of $H^{1/2}T$. Thus $A = B$ and since $H$ is assumed to be nonsingular this implies that $S = T$. It follows that the equation $THT = K$ can have at most one positive solution.

If $THT = K$ with $T$ in $B(\mathcal{S})_+$ then

$$(H^{1/2}K H^{1/2})^{1/2} = (H^{1/2}T H^{1/2} H^{1/2} T H^{1/2})^{1/2} = H^{1/2} T H^{1/2} \leq \|T\| H.$$ 

Conversely, if $(H^{1/2}K H^{1/2})^{1/2} \leq a H$ for some $a > 0$ then $(H^{1/2}K H^{1/2})^{1/4} = a^{1/2} S H^{1/2}$ for some $S$ in $B(\mathcal{S})$ with $\|S\| \leq 1$. This follows from a well-known variation of the polar decomposition theorem: If $A*A \leq B*B$ define $S_0 x = Ay$ for any $x$ in $\mathcal{S}$ such that $x = By$. Then $S_0$ extends uniquely to an operator $S$ in $B(\mathcal{S})$ with $\|S\| \leq 1$ such that $A = SB$. Let $T = a S* S$. Then $0 \leq T \leq a I$ and

$$H^{1/2} T H^{1/2} \leq (H^{1/2} T H^{1/2})^{1/2} \leq (a H^{1/2} S* S H^{1/2})^{1/2} = H^{1/2} K H^{1/2}.$$ 

Since $H$ is nonsingular this implies that $THT = K$.

If $H$ is invertible then $I \leq H^{-1/2} H$ so that each operator in $B(\mathcal{S})_+$ is majorized by a suitable multiple of $H$. In this case the solution to the equation $THT = K$ is given by the formula $T = H^{-1/2} (H^{1/2} K H^{1/2})^{1/2} H^{-1/2}$.

Suppose now that $K \leq a^2 H$ for some $a > 0$. Then $H^{1/2} K H^{1/2} \leq a^2 H^2$. Since the square root function is operator monotone (see [1]) this implies that $(H^{1/2} K H^{1/2})^{1/2} \leq a H$ so that $THT = K$ from the above. This completes the proof of the theorem.

References


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