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THE INTERSECTION MULTIPLICITY OF COMPACT n-DIMENSIONAL METRIC SPACES

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ABSTRACT. It is shown that there is an integer $\mu(n)$ such that any compact n-dimensional metric space M has intersection multiplicity at most $\mu(n)$. That is, if $\mathcal{U}$ is an open cover of M, then there is an open cover $\mathcal{V}$ refining $\mathcal{U}$ such that any element of $\mathcal{V}$ can intersect at most $\mu(n)$ other elements of $\mathcal{V}$.

For any open cover $\mathcal{U}$ of a topological space X define $m(\mathcal{U})$ to be the maximum number of elements of $\mathcal{U}$ that any member of $\mathcal{U}$ can intersect. X is said to have intersection multiplicity at most m if and only if every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ covering X such that $m(\mathcal{V})\leqq m$. The intersection multiplicity of X is then the least integer m such that X has intersection multiplicity at most m and is denoted $m(X)$.

THEOREM. *For every integer n there is an integer $\mu(n)$ such that, for every compact n-dimensional metric space M, $m(M)\leqq\mu(n)$.*

This result is known [3, p. 301] for differentiable manifolds and follows from two lemmas, the first of which is quite clear.

LEMMA 1. *If A is a closed subset of X, then $m(A)\leqq m(X)$.*

LEMMA 2. *If I^p is the unit cube in Euclidean space R^p, then $m(I^p)\leqq 3^p$.*

PROOF. Consider the cell complex J consisting of all cubes in R^p whose vertices have integral coordinates and whose edges have unit length. Let K be a simplicial subdivision of J with no new vertices [**1**, p. 11]. Let $\mathcal{V}$ be the cover of R^p consisting of the open stars of vertices in K. Let v and v' be vertices of K such that their open stars intersect. Then v and v' are vertices of a simplex in K, and their coordinates can differ by at most the integer 1. If v is fixed, there are at most 3^p choices for v'. Hence, $m(\mathcal{V})\leqq 3^p$.

Now let $\mathcal{U}$ be an open cover of I^p. Then for a sufficiently large integer q the linear transformation $L(x)=x/q$, when restricted to a subcomplex of K, gives a triangulation of I^p fine enough so that the image under L of $\mathcal{V}$, when restricted to I^p, refines $\mathcal{U}$. Thus $m(I^p)\leqq 3^p$.

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PROOF OF THE THEOREM. We may embed M as a closed subset of I^{2n+1} [**2**, p. 148]. Set $\mu(n)=m(I^{2n+1})$.

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