THE DIOPHANTINE APPROXIMATION OF
CERTAIN CONTINUED FRACTIONS

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Abstract. Given a real number \( \alpha \) defined by
\[
\frac{1}{\varphi(1)} + \frac{1}{\varphi(2)} + \cdots,
\]
where \( \varphi \) is a function from the natural numbers to the rational numbers larger than or equal to one which satisfies certain restrictions on the growth of the numerators and denominators of the numbers \( \varphi(n) \), then a lower bound is found in terms of \( \varphi \) for the diophantine approximation of \( \alpha \).

We wish to consider the continued fraction
\[
\frac{1}{\varphi(1)} + \frac{1}{\varphi(2)} + \cdots,
\]
where \( \varphi(n) \) and an auxiliary function \( \psi(n) \) satisfy certain conditions. Suppose that \( \varphi(n) \) and \( \psi(n) \) are functions defined from the positive integers to, respectively, the positive rationals and the positive integers such that
\[
\begin{align*}
\text{(a)} & \quad 1 \leq \varphi(j) \quad \text{for } j = 1, 2, \ldots; \\
& \quad \lim_{n \to \infty} \prod_{j=1}^{n} \varphi(j) = \infty, \quad \text{and} \\
\text{(b)} & \quad \limsup_{n \to \infty} (\log(2))(1 + n/2) \left( \log \left( \prod_{j=1}^{n-1} \varphi(j) \right) \right)^{-1} = \eta \\
& \quad \text{for some } 0 \leq \eta < +\infty; \\
\text{(c)} & \quad 0 \leq \limsup_{n \to \infty} (\log(\varphi(n + 1))) \left( \log \left( \prod_{j=1}^{n} \varphi(j) \right) \right)^{-1} = \delta < +\infty; \\
\text{(d)} & \quad \text{all } \psi(n) \left( \prod_{j \in S} \varphi(j) \right) \in Z \text{ if } S \subset \{1, \ldots, n\}; \quad \text{and} \\
\text{(e)} & \quad \limsup_{n \to \infty} (\log(\psi(n))) \left( \log \left( \prod_{j=1}^{n} \varphi(j) \right) \right)^{-1} = \gamma < 1.
\end{align*}
\]

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(Since each \( \varphi(j) > 0 \) and \( \sum_{j=1}^{\infty} \varphi(j) \geq \sum_{j=1}^{\infty} 1 = +\infty \), it follows immediately, from [1], that the continued fraction

\[
\frac{1}{\varphi(1)} + \frac{1}{\varphi(2)} + \ldots + \frac{1}{\varphi(n)} + \ldots
\]

converges to a positive real number \( \alpha \).)

**Theorem.** For every \( \varepsilon > 0 \), there exists a \( c(\varepsilon) > 0 \) such that, for every pair of positive integers \( p \) and \( q \),

\[
|\alpha - \frac{pq}{n}| > c(\varepsilon)q^{-(1+\delta+\varepsilon)}
\]

where \( \delta = (1+\varepsilon)(1+\eta+\varepsilon)(1-\gamma)^{-1} \).

**Examples.** We note that \( \varphi(n) = n \) and \( \psi(n) = 1 \) satisfy (a)–(e), as do \( \varphi(n) = (ts^{-1})^n \) and \( \psi(n) = s^{n(n+1)/2} \) satisfy (a)–(e) where \( p_\delta(x) \in Z^+[x] \) and \( p_\delta(x) \) has degree \( d \geq 1 \) in \( x \). One may generalize this to see that, where \( d_j, t_j, \) and \( s_j \), for \( 1 \leq j \leq m \), are positive integers satisfying

\[
\prod_{j=1}^{m} d_j / s_j > \left( \prod_{j=1}^{m} s_j \right)^2,
\]

where \( \varphi(n) = p_{d_1 \ldots d_m}(n, q_1^n, \ldots, q_m^n) \) is a polynomial in \( n \), \( q_1^n = (t_1 s_1^{-1})^n, \ldots, q_m^n = (t_m s_m^{-1})^n \) with positive integral coefficients and degrees \( d_0, \ldots, d_m \), respectively, of the form a nonzero polynomial in \( n \) times \( q_1^{d_1} \ldots q_m^{d_m} \) plus terms of lower degree in some \( q_j^n \), and where \( \psi(n) = \prod_{j=1}^{m} s_j^{n(n+1)/2} \) then hypotheses (a)–(e) of the Theorem are satisfied also.

One may drop the assumption of positive coefficients in the \( p_\delta(x) \) above. To see this, note first that the dominant term in each \( \varphi(n) \) forces it to be of constant sign for large enough \( n \). Then, for some positive integer \( N \), the hypotheses of the Theorem are satisfied upon substituting

\[
\beta = \frac{1}{|\varphi(N)| + |\psi(N + 1)|} \ldots
\]

The \((N+j)\)th convergent of the original continued fraction for \( \alpha \) is given by \((Ac_j + B)(Cc_j + D)^{-1} \), for integers \( A, B, C, \) and \( D \) with \(|c_j| \neq 0 \), where \( c_j \) is the \( j \)th convergent of the continued fraction above giving \( \beta \) and \( AC^{-1} \) and \( BD^{-1} \) are, respectively, the \( N \)th and \((N-1)\)st convergents of \( \alpha \). Since \( \beta \) is irrational (by (1)) eventually each \((Ac_j + B)(Cc_j + D)^{-1}\) is defined and, as \( j \rightarrow \infty \), these convergents approach \((A\beta + B)(C\beta + D)^{-1} = T(\beta)\), where \( T(z) = (Az + B)(Cz + D)^{-1} \).
If \(|\beta - T^{-1}(pq^{-1})| < |\beta - DC^{-1}|\) we have, by the law of the mean, that
\(|T(\beta) - pq^{-1}| = |T(\beta) - T(T^{-1}(pq^{-1}))| = |T'(\xi)| \cdot |\beta - T^{-1}(pq^{-1})|\)
where \(\xi\) is some point lying between \(\beta\) and \(T^{-1}(pq^{-1})\). By the continuity of \(T(z)\) at the
irrational point \(\beta\), we see the continuity of \(T^{-1}(z)\) at \(T(\beta)\) and this latter
property implies that, by requiring \(|T(\beta) - pq^{-1}|\) to be sufficiently small,
we can guarantee \(|\beta - T^{-1}(pq^{-1})| < |\beta - DC^{-1}|\). Thus either

(i) \(|T(\beta) - pq^{-1}| \geq K_1 \cdot |\beta - T^{-1}(pq^{-1})|\)

for some constant \(K_1 > 0\) independent of \(p\) and \(q\) or

(ii) \(|T(\beta) - pq^{-1}| > K_2 > 0\)

for some constant \(K_2\) independent of \(p\) and \(q\). One may conclude, from (1)
with \(\alpha = \beta\) and the alternatives (i) and (ii) above, that (1) holds with
\(\alpha = T(\beta)\), for a \(0 < c_1(e) \leq c(e)\) replacing \(c(e)\) in (1); since, if \(c_1(e) < K_2\),
the (ii) implies that (1) holds and case (i) says that \(|T(\beta) - pq^{-1}| \geq K_1 \cdot |\beta - T^{-1}(pq^{-1})|\),
which is larger than \(c_1(e)q^{-(1+8+e)}\) for some \(0 < c_1(e) \leq \min\{c(e), K_2\}\)
independent of \(p\) and \(q\).

In all of the above examples, \(\delta\) was zero. To see a case in which \(\delta > 0\)
set \(\varphi(n) = (5/2)^{2n}\) and \(\psi(n) = \frac{1}{2}(2^n+1)\). One could generalize along the lines
above and build up more complicated examples from this last example.

The author was led to consider the present problem after obtaining in
[2], by different but related methods, a lower bound on the simultaneous
diophantine approximation of the real number
\[
\frac{1}{z + \frac{z}{q + \frac{z}{q + \frac{z}{q + \ldots + \frac{z}{q + \frac{z}{q + q + q^2 + q^2 + \ldots}}}}}}
\]
where \(q\) denotes an integer, \(z\) denotes a rational number, \(|q| > 1\), and
\(|z| > 0\). (For \(z = 1\), the above number equals
\[
\prod_{n=0}^{\infty} (1 - q^{-(5n+2)})(1 - q^{-(5n+1)})(1 - q^{-(5n+3)})(1 - q^{-(5n+4)})^{-1},
\]
as was shown by Ramanujan.) Since we may rewrite this real number as
\[
1 + \frac{z}{q + \frac{z}{q + \frac{z}{q + \frac{z}{q + \ldots}}}}
\]
it follows that the present theory applies at \(z = 1\). In each case, we obtain
inequality (1) with \(\theta = 1\).

Proof of the Theorem. Since \(\sum_{n=2}^{\infty} \varphi(j) = + \infty\) we see that the con-
tinued fraction with these partial quotients converges to a positive real
number, \( \alpha \). Now \( \alpha = (\varphi(1) + \alpha')^{-1} \) so we have \( \alpha < (\varphi(1))^{-1} \). Using induction set \( a_0 = 1, \ a_1 = a_2 = \alpha < (\varphi(1))^{-1}, \ a_2 = a_4(\alpha(4))^{-1} - \varphi(1)) < (\varphi(1))^{-1} \), \( \cdots \), \( a_n = a_{n-1}(\alpha(4) - \varphi(n))^{-1} - \varphi(n-1)) < (\varphi(1))^{-1} \). Note that no \( a_j \) above is zero since each \( a_j \) is the product of nonzero real numbers. Also for \( n = 2, 3, \cdots \),

\[
a_n = -\varphi(n-1)a_{n-1} + a_{n-2}.
\]

Using (2) repeatedly, we may write each \( \psi(n-1)a_n \) as a linear form \( L_n = A_n a_1 + B_n a_0 \) for integers \( A_n \) and \( B_n \) satisfying easily calculable upper bounds on their absolute values. We find that

\[
|B_n| \leq \sum_{0 \leq k \leq n/2} \binom{n-k}{k} \left( \prod_{j=1}^{n-1} \varphi(j) \right) \varphi(n-1),
\]

where \( \binom{n-k}{k} < 2^{n-k} \). Thus \( B_n < 2^{1+n/2} \left( \prod_{j=1}^{n-1} \varphi(j) \right) \varphi(n-1) \). Similarly,

\[
|A_n| \leq \sum_{0 \leq k \leq n/2} \binom{n-1-k}{k} \left( \prod_{j=1}^{n-1} \varphi(j) \right) \varphi(n-1) < 2^{1+n/2} \left( \prod_{j=1}^{n-1} \varphi(j) \right) \varphi(n-1).
\]

Using hypotheses (b) and (e), for every \( \epsilon > 0 \),

\[
\max\{|A_n|, |B_n|\} < \left( \prod_{j=1}^{n-1} \varphi(j) \right)^{(1+\varphi(\gamma+\epsilon))}
\]

if \( n \) is sufficiently large. Under these same conditions we have, using (c), that

\[
\max\{|A_n|, |A_{n+1}|, |B_n|, |B_{n+1}|\} < \left( \prod_{j=1}^{n} \varphi(j) \right)^{(1+\varphi(\gamma+\epsilon))}.
\]

Also, if \( n \) is sufficiently large,

\[
\max\{|L_n|, |L_{n+1}|\} < \left( \prod_{j=1}^{n} \varphi(j) \right)^{-(1-\gamma+\epsilon)}.
\]

From (2), we may write

\[
\begin{pmatrix}
  a_n \\
  a_{n+1}
\end{pmatrix} =
\begin{pmatrix}
  0 & 1 \\
  1 & -\varphi(n)
\end{pmatrix}
\begin{pmatrix}
  a_{n-1} \\
  a_n
\end{pmatrix}.
\]

The entries of

\[
\begin{pmatrix}
  0 & 1 \\
  1 & -\varphi(n)
\end{pmatrix} \begin{pmatrix}
  0 & 1 \\
  1 & -\varphi(n-1)
\end{pmatrix} \cdots \begin{pmatrix}
  0 & 1 \\
  1 & -\varphi(1)
\end{pmatrix}
\]
must be proportional to those of
\[
\begin{pmatrix}
A_n & B_n \\
A_{n+1} & B_{n+1}
\end{pmatrix},
\]
therefore
\[
\begin{vmatrix}
A_n & B_n \\
A_{n+1} & B_{n+1}
\end{vmatrix} \neq 0.
\]

For each real \(1 \leq x < \infty\) set
\[
f(x) = \left( \prod_{j=1}^{[x]} \varphi(j) \right) (\varphi([x] + 1))^{x-[x]}^\theta_1
\]
where \([x]\) denotes the greatest integer function and \(\theta_1 = (1 + \delta)(1 + \gamma + \epsilon) = (1 - \gamma)^{-1}\). Note that \(f(x)\) is monotone increasing, and that \(f(x)\) takes the nonnegative reals onto \([\varphi(1), +\infty)\), since \(\prod_{j=1}^{\infty} \varphi(j) = \infty\). For \(n = 1, \ldots\) set
\[
A_n = \begin{pmatrix}
A_n & B_n \\
A_{n+1} & B_{n+1}
\end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 \\ a \end{pmatrix}.
\]

Let \(|\text{matrix}|\) denote the maximum of the absolute values of its entries. Then given \(0 < \epsilon < 1\), if \(n_1\) is sufficiently large we have, for all \(n \geq n_1\),

(i) \(|A_n| \leq (f(n))^{1+\epsilon/r}\) from (3),

(ii) \(|A_n V|^2 \leq (f(n))^{-\epsilon^2 (1-\epsilon/r)}\), from (4), where \(\theta_2 = (1 + \gamma + \epsilon)(1 - \gamma)^{-1}\), and

(iii) \(f(n) \leq (f(n-1))^{(1+\delta)(1+\epsilon/r)}\) from (c).

We shall next state a lemma from which we will be able to conclude the present Theorem. Then we shall conclude this paper with a proof of the Lemma.

Suppose that for some positive integer \(t\) we have a sequence \(A_n\) of \(t\) by \(1\) nonsingular matrices over the Gaussian integers and a \(t\) by 1 matrix \(V\) with complex entries. Let \(f(n)\) be a monotone increasing function from \([1, +\infty)\) onto \([c, +\infty)\) for some \(c \geq 1\). Let \(0 < \epsilon < +\infty\), \(0 < \gamma \leq +\infty\), and \((1 + \gamma)^2(1 - \gamma)^{-1} < 1 + \epsilon\). Suppose that, for all nonnegative integers \(n \geq n_1\),

(i) \(|A_n| \leq (f(n))^{1+\epsilon/r}\),

(ii) \(|A_n V|^2 \leq (f(n))^{-\epsilon^2 (1-\epsilon/r)}\) for some \(\Lambda > 0\), and

(iii) \(f(n) \leq (f(n-1))^{(1+\delta)(1+\epsilon/r)}\).

Let \(q\) denote a nonzero Gaussian integer and \(P\) denote a general \(t\) by 1 matrix of Gaussian integers with not all entries zero.

**Lemma.** If \(|q| > \frac{1}{2}(f(n_1))^{(1-\epsilon/r)}\) then \(|V - PQ^{-1}| \geq \left(\frac{1}{t} \right)^{1/2} |2q|^{-(1+\delta^2)/(1+\delta)}(1+\delta)/\Lambda\).
Note that if $0 < \epsilon < 1$ then 

$$(1 + \epsilon/5)^2(1 - \epsilon/5)^{-1} < (1 + \epsilon/4)^2(1 - \epsilon/4)^{-1} < 1 + 61(64)^{-1} \epsilon < 1 + \epsilon.$$ 

One may then apply the Lemma to our present situation with 

$$\Lambda = \theta_2^{-1} = (1 - \gamma)(1 + \eta + \gamma)^{-1} = (1 + \delta)^{-1}.$$ 

Thus we see that 

$$\| V - Pq^{-1} \| \geq \frac{1}{2} |2q|^{-(1 + (1 + \epsilon)/(\epsilon))},$$

where $V = (\xi)$. Setting $P = (\xi)$, where $p$ is an arbitrary nonnegative integer, and letting $q$ be an arbitrary positive integer, we obtain 

$$|x - pq^{-1}| > q^{-(1 + (1 + \epsilon)/(\epsilon))}$$

if $q$ is sufficiently large. Since (6) would be impossible if $x$ were rational, we see that there must exist a $c(\epsilon) > 0$ such that $|x - pq^{-1}| > c(\epsilon)q^{-(1 + \theta + \epsilon)}$ if $q \geq 1$. This proves our Theorem, assuming the Lemma.

**Proof of the Lemma.** For each nonnegative integer $n$,

$$\| \Delta_n (V - Pq^{-1}) \| \geq \| \Delta_n Pq^{-1} \| - \| \Delta_n V \|.$$ 

Then 

$$\| \Delta_n (V - Pq^{-1}) \| \geq |q|^{-1} - \| \Delta_n V \|,$$

since each entry in $P$ and $\Delta_n$ is a Gaussian integer, $P \neq 0$, and each $\Delta_n$ is nonsingular. We now choose $n$ to be the first integer such that $|2q| < (f(n))^{A(1 - \epsilon/r)}$. Since $(f(n))^{A(1 - \epsilon/r)} < |2q| < (f(n))^{A(1 - \epsilon/r)}$ and $f(n)$ is monotonically increasing we have $n > n_1$.

Therefore, we may use (ii) with (7) to conclude that

$$\| \Delta_n (V - Pq^{-1}) \| \geq |q|^{-1} - (f(n))^{-A(1 - \epsilon/r)}.$$ 

Since $|2q|^{-1} > (f(n))^{-A(1 - \epsilon/r)}$ we have

$$\| \Delta_n (V - Pq^{-1}) \| \geq |2q|^{-1}.$$ 

From our choice of $n$ and from hypothesis (iii), we see that

$$\exp((\log |2q|)A^{-1}(1 + \epsilon/r)^2(1 - \epsilon/r)^{-1}(1 + \delta))$$

$$\geq (f(n - 1))(1 + \epsilon/r)^2(1 + \delta) \geq (f(n))^{1 + \epsilon/r}.$$ 

Since $(1 + \epsilon/r)^2(1 - \epsilon/r)^{-1} < 1 + \epsilon$, we see that

$$\| \Delta_n (V - Pq^{-1}) \| \geq |2q|^{1 + \epsilon/r}.$$ 

From (8), hypothesis (i), and (9), we conclude that

$$\| V - Pq^{-1} \| \geq |2q|^{-1} |\Delta_n|^{-1} \geq |2q|^{-1} (f(n))^{-(1 + \epsilon/r)}$$

$$\geq |2q|^{-(1 + (1 + \epsilon)/(1 + \delta))}.$$ 

This proves the Lemma.
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