ON THE COMPLETE INTEGRAL CLOSURE OF A DOMAIN

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Abstract. For a given positive integer \( n \), a semivaluation domain \( D_n \) is constructed so that the complete integral closure has to be applied successively exactly \( n \) times before obtaining a completely integrally closed domain. Letting \( G_n \) be the group of divisibility of \( D_n \), we set \( G = \sum \mathbb{Z} G_n \), the cardinal sum of the groups \( G_n \). It is concluded that the semivaluation domain \( D \) having \( G \) as its group of divisibility is a Bezout domain with the property that \( D \subset D^* \subset D^{**} \subset D^{***} \subset \cdots \) is a strictly ascending infinite chain, where \( D^* \) is the complete integral closure of \( D \).

By a classical theorem of Krull, if \( G \) is a totally ordered abelian group then there exists a valuation domain \( D \) having \( G \) as its group of divisibility. There is, in fact, a standard construction of such a domain \( D \) based on the group algebra \( k(G) \) over any field \( k \). This construction was generalized from the case that \( G \) is totally ordered to a lattice ordered group \( G \) by P. Jaffard in [2]. Thus if \( G \) is an abelian \( l \)-group, we have via Krull-Jaffard an integral domain \( D[G] \) whose group of divisibility is \( G \). J. Ohm observed in [3] that this semivaluation domain \( D[G] \) is always a Bezout domain. The terminology "semivaluation domain" was established in [4].

The complete integral closure of a domain \( D \) in its quotient field is denoted herein by \( C(D) \) (a domain \( R \) is completely integrally closed if \( R \) contains each element \( x \) of its quotient field for which there exists an element \( d \neq 0 \) in \( R \) such that \( dx^n \in R \) for all \( n \geq 0 \)). More generally, for any positive integer \( n \), let \( C^n(D) = C(C^{n-1}(D)) \). In [1], W. Heinzer gave an example of a lattice ordered abelian group \( G \) such that \( C^2(D[G]) \neq C(D[G]) \). Thus, \( D[G] \) in Heinzer’s example is a Bezout domain whose complete integral closure is not completely integrally closed. In Heinzer’s example, \( D[G] \) has infinite dimension, but P. Sheldon [5] has refined the example so that \( D[G] \) has dimension two.

Received by the editors December 2, 1971.  
AMS 1970 subject classifications. Primary 13B20; Secondary 06A60.  
Key words and phrases. Complete integral closure, semivaluation, Bezout domain, group of divisibility, lattice ordered group.  
1 This research was supported in part by NSF Grant GP-29025.
Although we have examples where $C^2(D[G]) \neq C(D[G])$, it has not been known whether or not the inequality can persist between $C^n(D[G])$ and $C^{n+2}(D[G])$ for $n > 1$. In particular, we have the question concerning the complete integral closure of $C^2(D[G])$. In the examples of Heinzer and Sheldon, $C^2(D[G])$ is completely integrally closed, that is, $C^2(D[G]) = C^3(D[G])$. However, our purpose is to show that there exists a Bezout domain $D$ such that the operation of taking successively the complete integral closure beginning with $D$ does not stabilize in a finite number of steps. Since the domain $D$ with the desired properties turns out to be $D[G]$ for some $l$-group $G$, our interest is focused on $G$.

The desired lattice ordered group will be built in stages. The first component is closely related to Heinzer's example in [1]. Let $I \oplus J$ denote the lexicographic sum of two copies of the integers; $(i,j) \geq 0$ if $i \geq 1$ or if $i = 0$ and $j \geq 0$. Let $G_1$ denote the collection of all functions $f$ from the set $N$ of nonzero integers into $I \oplus J$ that satisfy the following conditions. If $f(n) = (i(n), j(n))$, then

1. $i(n) = 0$ whenever $n$ is negative.
2. $i(n) = 0$ for all but a finite number of positive $n$.
3. $j(n) = 0$ for all but a finite number of negative $n$, but there is no condition on $j(n)$ for positive $n$.

The set $G_1$ is a group with respect to pointwise addition. Furthermore, if we make $f \geq 0$ in $G_1$ if and only if $f(n) \geq 0$ for each $n \in N$, then $G_1$ is endowed with a lattice order. Before we describe $G_2$, two convex $l$-subgroups of $G_1$ are introduced. Let

$A = \{ f \in G_1 : f(n) = (i(n), j(n)), \text{ where } j(n) = 0 \}
\quad \text{for all but a finite number of } n \in N\}$,

and let

$K = \{ f \in G_1 : f(n) = (i(n), j(n)), \text{ where } i(n) = 0 \}
\quad \text{for all } n, j(n) = 0 \text{ if } n < 0, \text{ and } j(n) = 0 \}
\quad \text{for all but a finite number of positive } n\}$.

We now proceed to the construction of $G_2$. Let $P = P_1 + P_2 + \cdots + P_k + \cdots$ be a partition of the positive integers into an infinite number of infinite disjoint subsets $P_k$. Define the $l$-subgroup $G_2$ of $G_1$ by

$G_2 = \{ f \in G_1 : f(n) = (i(n), j(n)), \text{ then } (1) j(n) \}
\quad \text{is constant on } P_k \text{ for all but a finite number of } k, \text{ and (2) there exists, for each } k, \text{ integers } r \text{ and } s \text{ such that } j(n) = rn + s \}
\quad \text{for all but a finite number of } n \text{ in } P_k\}$.

One of the most interesting features of $G_2$ is that $G_2/K$ is lattice isomorphic to $G_1$ under a mapping $\phi: G_1 \rightarrow G_2/K$ defined as follows. If $f \in G_1$ and if
If \( f(n) = (i(n), j(n)) \), then \( \phi(f) = g + K \), where \( g(n) = (\alpha(n), \beta(n)) \) with \( \alpha(n) \) and \( \beta(n) \) being defined by:

- \( \alpha(n) = 0 \) if \( n \) is negative,
- \( \alpha(n) = j(-2n) \) if \( n \) is positive,
- \( \beta(n) = j(2n + 1) \) if \( n \) is negative,
- \( \beta(n) = ni(k) + j(k) \) if \( n \in \mathbb{P}_k \).

It is routine to verify that \( \phi \) is an isomorphism from \( G_1 \) onto \( G_2/K \). In order to show that \( \phi \) is an \( l \)-isomorphism, it suffices to show that \( \phi(f) \geq 0 \) if and only if \( f \geq 0 \). In this connection, recall that \( \phi(f) = g + K \geq 0 \) in \( G_2/K \) if and only if \( g + k \geq 0 \) in \( G_2 \) for some \( k \in K \). Now suppose that \( f \geq 0 \) in \( G_1 \). Then \( f(n) = (i(n), j(n)) \geq 0 \) in \( I \oplus J \) for each nonzero integer \( n \). It follows that \( i(n) \geq 0 \) for all \( n \in N \) and that \( j(n) \geq 0 \) for all negative \( n \) since \( i(n) = 0 \) if \( n < 0 \). Furthermore, if \( j(n) < 0 \) for a positive \( n \), then \( i(n) > 0 \). Thus if \( \beta(n) = ni(k) + j(k) < 0 \) for \( n \in \mathbb{P}_k \), then \( i(k) \neq 0 \). Therefore, \( i(k) > 0 \) and \( \beta(n) \geq 0 \) for all but a finite number of \( n \in \mathbb{P}_k \). Since \( i(k) = 0 \) for all but a finite number of \( k \), we conclude that \( \beta(n) < 0 \) for at most a finite number of positive \( n \).

Since \( K \) absorbs the components \( \beta(n) \) for a finite number of positive \( n \), it follows that \( \phi(f) = g + K = (\alpha(n), \beta(n)) + K \geq 0 \). The argument that \( \phi(f) \geq 0 \) implies that \( f \geq 0 \) follows from a similar but somewhat simpler analysis.

It is important to observe that \( \phi(A) \equiv A/K \), where \( A \) and \( K \) are the subgroups of \( G_1 \) introduced earlier. In particular, \( A \subseteq G_2 \). Letting \( \beta : G_2 \twoheadrightarrow G_2/K \xrightarrow{\phi^{-1}} G_1 \) be the composition of the natural map \( G_2 \to G_2/K \) and the \( l \)-isomorphism \( \phi^{-1} \), we have the exact sequence

\[
\begin{array}{ccc}
i & K & \longrightarrow \ G_2 & \beta \\
& & \longrightarrow & G_1,
\end{array}
\]

where \( i \) is the inclusion map and \( \beta \) is an \( l \)-homomorphism. We define \( G_n \) for \( n \geq 3 \) inductively by \( G_{n+1} = \beta^{-1}(G_n) \), the complete inverse image of \( G_n \) under \( \beta \). Since \( \beta \) is an \( l \)-homomorphism, \( G_{n+1} \) is an \( l \)-subgroup of \( G_n \) and \( G_{n+1}/K \) is \( l \)-isomorphic to \( G_n \). Furthermore, since \( \phi(A) \equiv A/K \), we see that \( \beta(A) \subseteq A \). Hence \( A \subseteq G_n \), for each \( n \), inductively.

Recall that an element \( b \geq 0 \) in any lattice ordered abelian group \( G \) is said to be a bounded element of \( G \) if there exists \( g \in G \) such that \( nb \leq g \) for each positive integer \( n \). The bounded elements of \( G \) form a convex subsemigroup of \( G \), and the group

\[ B(G) = \{ x : x = b - c \text{ where } b \text{ and } c \text{ are bounded in } G \} \]

is an ideal of \( G \). As is well known, there is a connection between bounded elements and the complete integral closure. If the Bezout domain \( D \) has \( G \) for its group of divisibility, then its complete integral closure \( C(D) \) has
G/B(G) for its group of divisibility. In particular, if G is any abelian l-group and D=D[G] is the semivaluation domain associated with G, then C(D) has G/B(G) for its group of divisibility. In view of this, we are especially interested in B(Gn) for the l-groups Gn that we have constructed. A simple inspection reveals that B(G1)=K=B(A), and therefore B(Gn)=K for all n since G1≥Gn≥A.

We have essentially established our first main result.

**Theorem 1.** For any nonnegative integer n, there exists a semivaluation domain D[Gn] such that

\[ C(D[Gn]) \subset C^2(D[Gn]) \subset \cdots \subset C^{n+1}(D[Gn]) = C^{n+2}(D[Gn]) = \cdots. \]

**Proof.** Suitable examples G0, for n=0, are in abundance. For example, we can take G0=A≤G1. If n>0, we shall show that the l-group Gn constructed above satisfies the condition of the theorem. We have observed that G(Bx)=K, but it is easy to verify that B(Gx/K)≠0, whereas B((Gx/K)/B(Gx/K))=0. We conclude that C(D[Gx])⊆C^2(D[Gx])=C^3(D[Gx]). The theorem now follows inductively from the l-isomorphism G_{n+1}/K=G_n and the equation B(G_{n+1})=K.

We see from Theorem 1 that, for any prescribed positive integer n, there exists a semivaluation domain D such that the complete integral closure has to be applied exactly n times before obtaining a completely integrally closed domain. The next theorem, however, shows that there exists a semivaluation domain D such that C^n(D) is not completely integrally closed for any n.

**Theorem 2.** There exists a (Bezout) semivaluation domain D such that the infinite sequence

\[ D \subset C(D) \subset C^2(D) \subset \cdots \subset C^n(D) \subset \cdots \]

is strictly ascending.

**Proof.** Choose the l-groups G_n so that they satisfy the conditions of Theorem 1. Let G=\( \sum_{n \geq 0} G_n \) be the (small) cardinal sum of the groups G_n. Letting Ĉ(G)=G/B(G) and letting Ĉk+1=ĈĈk, we see that

\[ Ĉ^k(G) = \sum_{n \geq 0} Ĉ^k(G_n) \]

since Ĉ commutes with cardinal sums. Since Ĉk(G_n) is different from zero if k<n+1, Ĉk(G) is different from zero for all n. Hence Ĉk(D[G]) is not completely integrally closed no matter what k is.
REFERENCES


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