PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, Number 1, November 1972

ON THE COMPLETE INTEGRAL CLOSURE OF A DOMAIN

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ABSTRACT. For a given positive integer n, a semivaluation domain D_n is constructed so that the complete integral closure has to be applied successively exactly n times before obtaining a completely integrally closed domain. Letting G_n be the group of divisibility of D_n, we set $G=\Sigma \boxplus G_n$, the cardinal sum of the groups G_n. It is concluded that the semivaluation domain D having G as its group of divisibility is a Bezout domain with the property that $D\subset D^*\subset D^{**}\subset D^{***}\subset\cdots$ is a strictly ascending infinite chain, where D^* is the complete integral closure of D.

By a classical theorem of Krull, if G is a totally ordered abelian group then there exists a valuation domain D having G as its group of divisibility. There is, in fact, a standard construction of such a domain D based on the group algebra $k(G)$ over any field k. This construction was generalized from the case that G is totally ordered to a lattice ordered group G by P. Jaffard in [2]. Thus if G is an abelian l-group, we have via Krull-Jaffard an integral domain $D[G]$ whose group of divisibility is G. J. Ohm observed in [3] that this semivaluation domain $D[G]$ is always a Bezout domain. The terminology "semivaluation domain" was established in [4].

The complete integral closure of a domain D in its quotient field is denoted herein by $C(D)$ (a domain R is *completely integrally closed* if R contains each element x of its quotient field for which there exists an element $d\neq 0$ in R such that $dx^n \in R$ for all $n\geqq 0$). More generally, for any positive integer n, let $C^n(D)=C(C^{n-1}(D))$. In [1], W. Heinzer gave an example of a lattice ordered abelian group G such that $C^2(D[G])\neq C(D[G])$. Thus, $D[G]$ in Heinzer's example is a Bezout domain whose complete integral closure is not completely integrally closed. In Heinzer's example, $D[G]$ has infinite dimension, but P. Sheldon [5] has refined the example so that $D[G]$ has dimension two.

Received by the editors December 2, 1971.

AMS 1970 *subject classifications*. Primary 13B20; Secondary 06A60.

Key words and phrases. Complete integral closure, semivaluation, Bezout domain, group of divisibility, lattice ordered group.

[1] This research was supported in part by NSF Grant GP-29025.

Although we have examples where $C^2(D[G]) \neq C(D[G])$, it has not been known whether or not the inequality can persist between $C^n(D[G])$ and $C^{n+1}(D[G])$ for $n>1$. In particular, we have the question concerning the complete integral closure of $C^2(D[G])$. In the examples of Heinzer and Sheldon, $C^2(D[G])$ is completely integrally closed, that is, $C^2(D[G])= C^3(D[G])$. However, our purpose is to show that there exists a Bezout domain D such that the operation of taking successively the complete integral closure beginning with D does not stabilize in a finite number of steps. Since the domain D with the desired properties turns out to be $D[G]$ for some l-group G, our interest is focused on G.

The desired lattice ordered group will be built in stages. The first component is closely related to Heinzer's example in [**1**]. Let $I \oplus J$ denote the lexicographic sum of two copies of the integers; $(i,j) \geqq 0$ if $i \geqq 1$ or if $i=0$ and $j \geqq 0$. Let G_1 denote the collection of all functions f from the set N of nonzero integers into $I \oplus J$ that satisfy the following conditions. If $f(n)= (i(n), j(n))$, then

(1) $i(n)=0$ whenever n is negative.

(2) $i(n)=0$ for all but a finite number of positive n.

(3) $j(n)=0$ for all but a finite number of negative n, but there is no condition on $j(n)$ for positive n.

The set G_1 is a group with respect to pointwise addition. Furthermore, if we make $f \geqq 0$ in G_1 if and only if $f(n) \geqq 0$ for each $n \in N$, then G_1 is endowed with a lattice order. Before we describe G_2, two convex l-subgroups of G_1 are introduced. Let

$$A = \{f \in G_1 : f(n) = (i(n), j(n)), \text{ where } j(n) = 0 \text{ for all but a finite number of } n \in N\},$$

and let

$$K = \{f \in G_1 : f(n) = (i(n), j(n)), \text{ where } i(n) = 0 \text{ for all } n, j(n) = 0 \text{ if } n < 0, \text{ and } j(n) = 0 \text{ for all but a finite number of positive } n\}.$$

We now proceed to the construction of G_2. Let $P = P_1 + P_2 + \cdots + P_k + \cdots$ be a partition of the positive integers into an infinite number of infinite disjoint subsets P_k. Define the l-subgroup G_2 of G_1 by

$$G_2 = \{f \in G_1 : \text{if } f(n) = (i(n), j(n)), \text{ then (1) } j(n) \text{ is constant on } P_k \text{ for all but a finite number of } k, \text{ and (2) there exists, for each } k, \text{ integers } r \text{ and } s \text{ such that } j(n) = rn + s \text{ for all but a finite number of } n \text{ in } P_k\}.$$

One of the most interesting features of G_2 is that G_2/K is lattice isomorphic to G_1 under a mapping $\phi: G_1 \to G_2/K$ defined as follows. If $f \in G_1$ and if

$f(n)=(i(n),j(n))$, then $\phi(f)=g+K$, where $g(n)=(\alpha(n),\beta(n))$ with $\alpha(n)$ and $\beta(n)$ being defined by:

$$\begin{aligned} \alpha(n) &= 0 && \text{if } n \text{ is negative,} \\ \alpha(n) &= j(-2n) && \text{if } n \text{ is positive,} \\ \beta(n) &= j(2n+1) && \text{if } n \text{ is negative,} \\ \beta(n) &= ni(k)+j(k) && \text{if } n \in P_k. \end{aligned}$$

It is routine to verify that ϕ is an isomorphism from G_1 onto G_2/K. In order to show that ϕ is an l-isomorphism, it suffices to show that $\phi(f)\geqq 0$ if and only if $f\geqq 0$. In this connection, recall that $\phi(f)=g+K\geqq 0$ in G_2/K if and only if $g+k\geqq 0$ in G_2 for *some* $k\in K$. Now suppose that $f\geqq 0$ in G_1. Then $f(n)=(i(n),j(n))\geqq 0$ in $I\oplus J$ for each nonzero integer n. It follows that $i(n)\geqq 0$ for all $n\in N$ and that $j(n)\geqq 0$ for all negative n since $i(n)=0$ if $n<0$. Furthermore, if $j(n)<0$ for a positive n, then $i(n)>0$. Thus if $\beta(n)=ni(k)+j(k)<0$ for $n\in P_k$, then $i(k)\neq 0$. Therefore, $i(k)>0$ and $\beta(n)\geqq 0$ for all but a finite number of $n\in P_k$. Since $i(k)=0$ for all but a finite number of k, we conclude that $\beta(n)<0$ for at most a finite number of positive n. Since K absorbs the components $\beta(n)$ for a finite number of positive n, it follows that $\phi(f)=g+K=(\alpha(n),\beta(n))+K\geqq 0$. The argument that $\phi(f)\geqq 0$ implies that $f\geqq 0$ follows from a similar but somewhat simpler analysis.

It is important to observe that $\phi(A)\supseteq A/K$, where A and K are the subgroups of G_1 introduced earlier. In particular, $A\subseteq G_2$. Letting $\beta: G_2\twoheadrightarrow G_2/K \rightarrowtail\!\!\!\twoheadrightarrow^{\phi^{-1}} G_1$ be the composition of the natural map $G_2\twoheadrightarrow G_2/K$ and the l-isomorphism ϕ^{-1}, we have the exact sequence

$$K \overset{i}{\rightarrowtail} G_2 \overset{\beta}{\twoheadrightarrow} G_1,$$

where i is the inclusion map and β is an l-homomorphism. We define G_n for $n\geqq 3$ inductively by $G_{n+1}=\beta^{-1}(G_n)$, the complete inverse image of G_n under β. Since β is an l-homomorphism, G_{n+1} is an l-subgroup of G_n and G_{n+1}/K is l-isomorphic to G_n. Furthermore, since $\phi(A)\supseteq A/K$, we see that $\beta(A)\subseteq A$. Hence $A\subseteq G_n$, for each n, inductively.

Recall that an element $b\geqq 0$ in any lattice ordered abelian group G is said to be a bounded element of G if there exists $g\in G$ such that $nb\leqq g$ for each positive integer n. The bounded elements of G form a convex subsemigroup of G, and the group

$$B(G) = \{x : x = b - c \text{ where } b \text{ and } c \text{ are bounded in } G\}$$

is an ideal of G. As is well known, there is a connection between bounded elements and the complete integral closure. If the Bezout domain D has G for its group of divisibility, then its complete integral closure $C(D)$ has

$G/B(G)$ for its group of divisibility. In particular, if G is any abelian l-group and $D=D[G]$ is the semivaluation domain associated with G, then $C(D)$ has $G/B(G)$ for its group of divisibility. In view of this, we are especially interested in $B(G_n)$ for the l-groups G_n that we have constructed. A simple inspection reveals that $B(G_1)=K=B(A)$, and therefore $B(G_n)=K$ for all n since $G_1 \supseteq G_n \supseteq A$.

We have essentially established our first main result.

THEOREM 1. *For any nonnegative integer n, there exists a semivaluation domain $D[G_n]$ such that*

$$C(D[G_n]) \subset C^2(D[G_n]) \subset \cdots \subset C^{n+1}(D[G_n]) = C^{n+2}(D[G_n]) = \cdots .$$

PROOF. Suitable examples G_0, for $n=0$, are in abundance. For example, we can take $G_0=A\subseteq G_1$. If $n>0$, we shall show that the l-group G_n constructed above satisfies the condition of the theorem. We have observed that $G(B_1)=K$, but it is easy to verify that $B(G_1/K)\neq 0$, whereas $B((G_1/K)/B(G_1/K))=0$. We conclude that $C(D[G_1])\subset C^2(D[G_1])= C^3(D[G_1])$. The theorem now follows inductively from the l-isomorphism $G_{n+1}/K\cong G_n$ and the equation $B(G_{n+1})=K$.

We see from Theorem 1 that, for any prescribed positive integer n, there exists a semivaluation domain D such that the complete integral closure has to be applied exactly n times before obtaining a completely integrally closed domain. The next theorem, however, shows that there exists a semivaluation domain D such that $C^n(D)$ is not completely integrally closed for any n.

THEOREM 2. *There exists a (Bezout) semivaluation domain D such that the infinite sequence*

$$D \subset C(D) \subset C^2(D) \subset \cdots \subset C^n(D) \subset \cdots$$

is strictly ascending.

PROOF. Choose the l-groups G_n so that they satisfy the conditions of Theorem 1. Let $G=\sum_{n\geq 0} \boxplus\, G_n$ be the (small) cardinal sum of the groups G_n. Letting $\bar{B}(G)=G/B(G)$ and letting $\bar{B}^{k+1}=\bar{B}\bar{B}^k$, we see that

$$\bar{B}^k(G) = \sum_{n\geq 0} \boxplus\, \bar{B}^k(G_n)$$

since $\bar{B}$ commutes with cardinal sums. Since $\bar{B}^k(G_n)$ is different from zero if $k<n+1$, $\bar{B}^k(G)$ is different from zero for all n. Hence $C^k(D[G])$ is not completely integrally closed no matter what k is.

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