AN ELEMENTARY PROOF OF STEINHAUS'S THEOREM

KARL STROMBERG

The following theorem was proved by H. Steinhaus [2] for the case that $G$ is the real line and $\lambda$ is Lebesgue measure. In the form stated here, it is due to André Weil [3, p. 50].

**Theorem.** Let $G$ be a locally compact group with identity $e$ and a left Haar measure $\lambda$. If $A$ is a $\lambda$-measurable subset of $G$ such that $0 < \lambda(A) < \infty$, then the set $AA^{-1} = \{yx^{-1} : x, y \in A\}$ has $e$ in its interior.

The customary proof of this well-known theorem uses the convolution product of functions on $G$. Our proof, which does not seem to be widely known, is shorter and more elementary, even for the real line, than any that we have seen. All references of the form (a,b) are to be found in [1].

**Proof.** Since $X$ is regular, we may suppose that $A$ is compact (11.32) and we may choose an open set $U \supset A$ such that $\lambda(U) < 2\lambda(A)$ (11.22). Next choose a neighborhood $V$ of $e$ such that $VA \subset U$ (4.10). We complete the proof by showing that $V \subset AA^{-1}$. Let $v \in V$. Then $vA \cap A \neq \emptyset$ because otherwise, since $vA \cup A \subset U$, we would have $\lambda(U) \geq \lambda(vA) + \lambda(A) = 2\lambda(A)$. Therefore, there exist $x, y \in A$ such that $vx = y$ and so $v = yx^{-1} \in AA^{-1}$.

**References**


Department of Mathematics, Kansas State University, Manhattan, Kansas 66502

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