ON THE COEFFICIENTS OF FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

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Abstract. Let $V_k$ be the class of normalised functions of bounded boundary rotation. For $f \in V_k$ define

$$M(r, f) = \max_{|z| = r} |f(z)|,$$

and let $L(r, f)$ denote the length of $f(|z| = r)$. Then if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, it is shown that (i) $2M(r, f) < L(r, f) \leq k \pi M(r, f)$, and (ii) $n^2 |a_n| \leq (3k/r^n) M(r, f')$, $n \geq 2$. The class $A_k$ of meromorphic functions of boundary rotation is also studied and estimates for the coefficients are given.

1. Introduction. For fixed $k \geq 2$, let $V_k$ denote the class of normalised functions of boundary rotation at most $k\pi$; that is, $f \in V_k$ if and only if $f$ is analytic in the open unit disc $\gamma, f'(z) \neq 0$ for $z \in \gamma, f(0) = 0, f'(0) = 1$, and $f$ maps $\gamma$ onto a domain with boundary rotation at most $k\pi$. Since the boundary rotation is the total variation of the argument of the boundary tangent vector (whenever such a tangent vector exists), we have (see e.g. [5]), with $z = re^{i\theta}$,

$$\int_0^{2\pi} \left| \frac{\text{Re}(zf'(z))'}{f'(z)} \right| d\theta \leq k\pi.$$

$V_2$ is the class of normalised convex functions, and it is well known that for $2 \leq k \leq 4$, $V_k$ contains only univalent functions.

Suppose $f \in V_k$ and has Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Then the problem $A_n(k) = \max |a_n|$ has been extensively studied, but remains largely unsolved. It is known that, for $k \geq 2$,

$$A_2(k) = k/2, \quad A_3(k) = (k^2 + 2)/6, \quad A_4(k) = (k^3 + 8k)/24,$$

and, for $n \geq 2$, that

$$|a_n| \leq c(k)n^{k/2 - 1}.$$

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where \( c(k) \) is a constant depending only upon \( k \). (4) was given in [4] with \( c(k) \to \infty \) as \( k \to \infty \) and in [8] with \( c(k) \to 0 \) as \( k \to \infty \). The function \( f_0 \in V_k \) defined for \( z \in \gamma \) by

\[
f_0(z) = \frac{1}{ek} \left[ \left( \frac{1 + \epsilon z}{1 - \epsilon z} \right)^{k/2} - 1 \right], \quad |\epsilon| = 1,
\]

shows that equality may occur in each of (3), and also that the index of \( n \) in (4) is best possible.

A class of functions closely related to \( V_k \) is the class \( \Lambda_k \) (\( k \geq 2 \)) of meromorphic functions defined as follows. The function \( g \), given by

\[
g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n,
\]

belongs to \( \Lambda_k \) if and only if \( g \) is analytic in \( \gamma' = \gamma \setminus \{0\} \), \( g'(z) \neq 0 \) for \( z \in \gamma' \) and \( g \) maps \( \gamma' \) onto a domain with boundary rotation at most \( k \pi \). For \( g \in \Lambda_k \), we have [7], with \( z = re^{i\theta} \),

\[
\int_0^{2\pi} \left| \frac{\operatorname{Re} (zg'(z))}{g(z)} \right| d\theta \leq k\pi.
\]

If \( g \in \Lambda_k \) is given by (5), then the problem \( B_n(k) = \max |b_n| \) was considered in [6]. It was shown that

\[
B_1(k) = k/2 \quad \text{and} \quad B_2(k) = k/6.
\]

It was also shown in [6] that, for \( n \geq 1 \), \( |b_n| \leq C(k)n^{k/2-3} \), for \( k \geq 2 \) and \( k \geq 4 \), where \( C(k) \to \infty \) as \( k \to \infty \).

Let \( M(r,f) = \max_{|z|=r} |f(z)| \) and let \( 0 < r < 1 \). The main purpose of this paper is to give estimates for the coefficients \( a_n \) and \( b_n \) in (2) and (5) in terms of \( M(r,f') \) and \( M(r,g') \) respectively. We shall also give an extremely simple proof of (4) with an improved constant \( c(k) \). The methods of the paper show also that, for all \( k \geq 2 \) and \( n \geq 2 \), \( |b_n| \leq C(k)n^{k/2-3} \) where \( C(k) \to 0 \) as \( k \to \infty \). This result improves the theorem given in [6], since the estimate is now valid for all \( k \geq 2 \), and since \( C(k) \to 0 \) as \( k \to \infty \).

2. Statement of results. For \( V_k \) we have:

**Theorem 1.** Let \( f \in V_k \) and be given by (2). Then, for any \( 0 < r < 1 \),

(i) \( 2M(r,f) < L(r) \leq k\pi M(r,f) \) where \( L(r) \) is the image of \( \gamma_r = \{ z : |z| = r \} \) under \( f \), and

(ii) \( n^2 |a_n| \leq (3k/r^{n-1})M(r,f'), \quad n \geq 2. \)

**Remark.** 1. A geometrical proof of (i) was given in [2] with a worse constant.
2. An example in [2] shows that the constant $k\pi$ in (i) is the best possible.

With the aid of this theorem we are able to prove the following corollaries:

**Corollary 1.** Let $f \in V_k$ and be given by (2). Then, for $n \geq 5$,

$$n |a_n| \leq e^{t/2}[(n/2)^{k/2} - 1].$$

For $n=2, 3, 4$ the inequalities (3) are better.

**Corollary 2.** Let $f \in V_k$ and be given by (2). Then, for $n \geq 2$,

$$n |a_n| \leq 2ke^2(A(1-1/n)/\pi)^{1/2},$$

where $A(r)$ is the area of $f(\gamma_r)$.

This last result was obtained in [1], with essentially the same constants. In [2] it was also shown that for a bounded function in $V_k$, $na_n=o(1)$ as $n \to \infty$. The following extends this result.

**Corollary 3.** Let $f \in V_k$ and be given by (2). Then if the area of $f(\gamma)$ is finite, $na_n=o(1)$ as $n \to \infty$, and the index of $n$ is best possible.

For $\Lambda_k$ we have

**Theorem 2.** Let $g \in \Lambda_k$ and be given by (5). Then, for any $0<r<1$ and any $n \geq 1$,

$$n^2 |b_n| \leq (4k/r^{n-1})M(r, g').$$

We also have

**Corollary 4.** Let $g \in \Lambda_k$ and be given by (5). Then, for $n \geq 2$,

$$|b_n| \leq (32e^{4k/2^{1/2}})n^{5/2-3}.$$

The function $g_0 \in \Lambda_k$, defined by

$$g_0(z) = -\frac{1}{z^2} \frac{(1 + z^2 - 2z(k - 2)/(k + 2))^{(k+2)/4}}{(1 - z)^{k-2}}.$$  

shows that the index of $n$ is best possible.

3. **Proofs of theorems.**

**Proof of Theorem 1.** (i) Write

$$L(r) = \int_0^{2\pi} |zf'(z)| \, d\theta = \int_0^{2\pi} zf'(z)\exp[-i \arg zf'(z)] \, d\theta.$$
Then integration by parts gives

\[ L(r) = \int_0^{2\pi} f(z) \exp[-i \arg zf'(z)] \partial_\phi(\arg zf'(z)) \, d\theta \leq k\pi M(r, f) \]

on using (1). The left hand inequality is trivial.

(ii) We shall use the method of Clunie and Pommerence [3], and shall need the following lemma.

**Lemma 1.** Let \( f \in V_k \) and, for \( n \geq 1 \), \( z = re^{i \theta} \), put

\[ J_n(r) = \frac{1}{2\pi} \int_0^{2\pi} (zf'(z)')' z^{n+1} \exp[-2i(\theta + \arg f'(z))] \, d\theta. \]

Then \( |J_n(r)| \leq 2kr^{n+1}M(r, f') \).

**Proof.** Let \( F(z) = (zf'(z))'/f'(z) \). Then \( (\partial/\partial \theta)(\theta + \arg f'(z)) = \Re F(z) \).

Integration by parts shows that

\[ J_n(r) = \frac{1}{\pi} \int_0^{2\pi} f_n(z) \exp[-2i(\theta + \arg f'(z))] \Re F(z) \, d\theta \]

where

\[ f_n(z) = \int_0^z \xi^n (\xi f'(\xi))' \, d\xi. \]

Again using integration by parts we have

\[ f_n(z) = z^{n+1}f'(z) - n \int_0^z \xi^n f'(\xi) \, d\xi, \]

and so

\[ |f_n(z)| \leq r^{n+1}M(r, f') + nM(r, f') \int_0^r t^n \, dt \leq 2r^{n+1}M(r, f'). \]

From (7) we now have

\[ |J_n(r)| \leq \frac{2r^{n+1}}{\pi} M(r, f') \int_0^{2\pi} |\Re F(z)| \, d\theta, \]

and Lemma 1 now follows on using (1).

We now prove (ii). With \( (zf'(z))' = f'(z)F(z) \) we write \( F(z) = 2 \Re F(z) - F(z)^* \), where * denotes complex conjugate. Then with \( z = re^{i \theta} \), we have

\[ n^2 a_n = \frac{1}{2\pi r^{n-1}} \int_0^{2\pi} (zf'(z))' \exp[-i(n - 1)\theta] \, d\theta \]

\[ = \frac{1}{\pi r^{n-1}} \int_0^{2\pi} f'(z) \Re F(z) \exp[-i(n - 1)\theta] \, d\theta \]

\[ - \frac{1}{2\pi r^{n-1}} \int_0^{2\pi} f'(z) F(z)^* \exp[-i(n - 1)\theta] \, d\theta. \]
Hence

\[ n^2 |a_n| \leq \frac{1}{\pi r^{n-1}} \int_0^{2\pi} |f'(z)| |\text{Re } F(z)| \, d\theta \]

\[ + \frac{1}{2\pi r^{n-1}} \left| \int_0^{2\pi} f'(z)^* F(z) \exp[i(n-1)\theta] \, d\theta \right| \]

\[ = P_1 + P_2, \quad \text{say,} \]

where in \( P_2 \) we have taken the complex conjugate.

From (1) we obtain at once \( P_1 \leq (k/r^{n-1}) M(r, f') \). Now

\[ f'(z)^* F(z) = (zf'(z))^* \exp[-2i \arg f'(z)], \]

and so

\[ P_2 = \frac{1}{2\pi r^{n-1}} \left| \int_0^{2\pi} (zf'(z))^* \exp[i(n+1)\theta] \exp[-2i(\theta + \arg f'(z))] \, d\theta \right| \]

\[ = (1/r^{2n}) |J_n(r)|. \]

Thus from Lemma 1, \( P_2 \leq (2k/r^{n-1}) M(r, f') \). Hence

\[ n^2 |a_n| \leq (3k/r^{n-1}) M(r, f'), \]

which proves (ii).

**Proof of Corollary 1.** It is well known [5] that for \( f \in V_k \),

\[ |f(re^{i\theta})| \leq \frac{1}{k} \left[ \left( \frac{1 + r^{k/2}}{1 - r} \right) - 1 \right] \]

With \( n \geq 5 \), choose \( r = 1 - 4/n \), then from Theorem 1 (i), with \( z = re^{i\theta} \),

\[ n |a_n| \leq \frac{1}{2\pi r^n} \int_0^{2\pi} |zf'(z)| \, d\theta \leq \frac{1}{2r^n} \left[ \left( \frac{1 + r^{k/2}}{1 - r} \right) - 1 \right] < \frac{e^{4r}}{2} \left[ \left( \frac{n}{2} \right)^{k/2} - 1 \right]. \]

**Proof of Corollary 2.** Note that

\[ rM(r, f') \leq \sum_{n=1}^{\infty} n |a_n|^2 r^n \]

\[ \leq \left( \sum_{n=1}^{\infty} n |a_n|^2 r^n \right)^{1/2} \left( \sum_{n=1}^{\infty} n r^n \right)^{1/2} \leq \left( \frac{A(r)}{\pi} \right)^{1/2} \frac{1}{1 - r}. \]

For \( n \geq 2 \), choose \( r = (1 - 1/n)^2 \), and the result follows at once from Theorem 1 (ii).

**Proof of Corollary 3.** This follows at once from Theorem 1 (ii) on noting that if the area of \( f(\gamma) \) is finite, then \( M(r, f') = o(1)/(1 - r) \) as \( r \to 1 \).

The function \( f_z : f_0(z) = (1/x)[1 - (1 - z)^x] \) for \( 0 < x < 1 \) is convex and bounded and shows that the index of \( n \) is best possible.

**Proof of Theorem 2.** We need a lemma analogous to Lemma 1.
**Lemma 2.** Let $g \in \Lambda_k$ and, for $n \geq 2$, $z = re^{i\theta}$, put

$$K_n(r) = \frac{1}{2\pi} \int_0^{2\pi} (zg'(z))^n z^{n+1} \exp[-2i(\theta + \arg g'(z))] \, d\theta.$$  

Then $|K_n(r)| \leq 3kr^{n+1}M(r, g')$.

**Proof.** Let $G(z) = (zg'(z))'/g'(z)$. Then $G$ is analytic in $\gamma'$ and $(\partial/\partial\theta)(\theta + \arg g'(z)) = \Re G(z)$. Integration by parts shows that

$$K_n(r) = \frac{1}{\pi} \int_0^{2\pi} g_n(z) \exp[-2i(\theta + \arg g'(z))] \Re G(z) \, d\theta$$

where $g_n(z) = \int_0^z \xi^n (\xi g'(\xi))' \, d\xi$ (Note: $n \geq 2$ assures regularity of the integrand at $\xi = 0$.)

Again using integration by parts,

$$g_n(z) = z^{n+1}g'(z) - n \int_0^z \xi^n g'(\xi) \, d\xi,$$

and so $|g_n(z)| \leq 3r^{n+1}M(r, g')$. Exactly as in Lemma 1 we now find that $|K_n(r)| \leq 3kr^{n+1}M(r, g')$, which proves Lemma 2.

We now prove Theorem 2. For $n \geq 2$ we have

$$n^2b_n = \frac{1}{\pi r^{n-1}} \int_0^{2\pi} (zg'(z))' \exp[-i(n - 1)\theta] \, d\theta$$

$$= \frac{1}{\pi r^{n-1}} \int_0^{2\pi} g'(z) \exp[-i(n - 1)\theta] \Re G(z) \, d\theta$$

$$- \frac{1}{\pi r^{n-1}} \int_0^{2\pi} g'(z) G(z)^* \exp[-i(n - 1)\theta] \, d\theta.$$  

Hence

$$n^2 |b_n| \leq \frac{1}{\pi r^{n-1}} \int_0^{2\pi} |g'(z)| |\Re G(z)| \, d\theta$$

$$+ \frac{1}{\pi r^{n-1}} \left| \int_0^{2\pi} g'(z)^* G(z) \exp[i(n - 1)\theta] \, d\theta \right|$$

$$= Q_1 + Q_2, \text{ say.}$$

As before, $Q_1 \leq (k/r^{n-1})M(r, g')$. Also,

$$Q_2 = \frac{1}{\pi r^{n-1}} \left| \int_0^{2\pi} (zg'(z)^')' \exp[i(n + 1)\theta] \exp[-2i(\theta + \arg g'(z))] \, d\theta \right|$$

$$= (1/r^2n) |K_n(r)| \leq (3k/r^{n-1})M(r, g').$$
by Lemma 2. Thus, for \( n \geq 2 \),
\[
n^2 |b_n| \leq (4k/r^{n-1})M(r, g').
\]
An elementary estimate using the Cauchy integral formula shows the above estimate is also true for \( n = 1 \), and so the proof is complete.

**Proof of Corollary 4.** In [6] it is shown that \( g \in \Lambda_k \) if and only if there exists \( f \in V_k \) with \( a_2 = 0 \) such that
\[
-z^2 g'(z) = 1/f'(z),
\]
and that
\[
M(r, g') \leq \frac{(1 + r)^{k/2+1}}{r^2(1 - r)^{k/2-1}}.
\]
We remark that (9) is certainly not sharp, but is sufficient for our purpose. Let \( n \geq 5 \) and choose \( r = 1 - 4/n \). Then from Theorem 2 and (9) we have
\[
|b_n| \leq (32k e^{4/2k})n^{k/2-3}.
\]
It remains only to show that this estimate is valid for \( n = 2, 3, 4 \). For \( n = 2 \), (10) follows since \( |b_2| \leq k/6 \). Using (8) together with the condition \( a_2 = 0 \), it is easily seen on equating coefficients that \( |b_3| \leq k^2/24 + k/12 \), which gives (10) for \( n = 3 \). Similarly one can obtain \( |b_4| \leq k^2/24 + k/20 \), which again gives (10) for \( n = 4 \).

**References**


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