A CONDITION FOR ANALYTIC STRUCTURE

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Abstract. Let \( X \) be a compact Hausdorff space, \( A \) a uniform algebra on \( X \), \( M \) the maximal ideal space of \( A \). Let \( f \in A \) and let \( W \) be a component of \( C \setminus f(X) \). Suppose that, for all \( \lambda \in W \), \( f^{-1}(\lambda)\) is at most countable. Then there is an open dense subset \( U \) of \( f^{-1}(W) \) which can be given the structure of a one-dimensional complex analytic manifold so that for all \( g \in A \), \( g \) is analytic on \( U \).

Let \( X \) be a compact Hausdorff space, let \( A \) be a uniform algebra on \( X \), and let \( M \) be the maximal ideal space of \( A \).

Theorem. Let \( f \in A \) and let \( W \) be a component of \( C \setminus f(X) \). Suppose that for all \( X \in W \), \( f^{-1}(X) = \{ x \in M \mid f(x) = X \} \) is at most countable. Then there is an open dense subset \( f^{-1}(W) \) which can be given the structure of a one-dimensional complex analytic manifold so that the functions in \( A \) become analytic there.

This theorem partially generalizes results for the case when \( f^{-1}(\lambda) \) is finite for all \( \lambda \) in a sufficiently large subset of \( W \), which are essentially contained in a paper of E. Bishop ([1], see Theorem 11.2 in [2]). A key result in the proof of this earlier result is also important in the proof of our theorem (see Lemma 13 in [1] or Theorem 10.7 in [2]).

Definition. Let \( p \in M \), and let \( \Phi \) be a continuous one-to-one map from \( \{ |z| < 1 \} \) into \( M \), with \( \Phi(0) = p \). The set \( \{ \Phi(z) \mid |z| < 1 \} \) is called an analytic disk through \( p \) if for all \( h \in A \), \( h \circ \Phi \) is analytic on \( \{ |z| < 1 \} \).

Lemma 1. Let \( f \in A \) and suppose that:

(a) \( |f| = 0 \) on \( X \);
(b) \( \exists \ p \in M \) with \( f(p) = 0 \);
(c) \( \exists \) a closed subset \( \Gamma_0 \) of \( \{ |z| = 1 \} \) having positive linear measure such that for each \( \lambda \in \Gamma_0 \) there is a unique point \( q \in X \) with \( f(q) = \lambda \).

Then \( f^{-1}(|z| < 1) \) is an analytic disk through \( p \).

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We also need the following elementary result (Lemma 11.1 in [2]).

**Lemma 2.** Let $f \in A$ and let $W$ be a component of $C \setminus f(X)$. Fix $\lambda \in W$. If $f$ takes on the value $\lambda$ on $M$, then $f$ takes on every value in $W$ on $M$.

One further result is needed.

**Lemma 3.** Let $f \in A$ and let $W$ be a component of $C \setminus f(X)$. Let $z \in W$, and suppose that $f^{-1}(z)$ is at most countable. Given a neighborhood $\emptyset$ of a point $x \in f^{-1}(z)$, there is a compact neighborhood $N$ of $x$ with the properties that:

(a) $N$ is $A$-convex, i.e., the maximal ideal space of $A|_{N}$ is $N$;
(b) $z \notin f(\partial N)$;
(c) $N \subseteq \emptyset \cap f^{-1}(W)$.

Furthermore, $f(N)$ is a neighborhood of $z$ in $C$.

**Proof of Lemma 3.** Choose $\varepsilon > 0$ and functions $g_1, g_2, \cdots, g_n \in A$ so that

$$N = \{ y \in M \mid |g_j(y) - g_j(x)| \leq \varepsilon, 1 \leq j \leq n \} \subseteq f^{-1}(W) \cap \emptyset.$$

Since $f^{-1}(z)$ is at most countable, we may adjust $\varepsilon$ so that, for all $y \in f^{-1}(z)$, $|g_j(y) - g_j(x)| \neq \varepsilon$, $j = 1, 2, \cdots, n$. Now if $y \in \partial N$, then $|g_j(y) - g_j(x)| = \varepsilon$ for some $j$, so $f^{-1}(z) \cap \partial N = \emptyset$ and it is clear that (a), (b) and (c) are satisfied.

By the local maximum modulus principle, the Shilov boundary of $A|_{N}$ is contained in $\partial N$ (note that $N$ does not meet the Shilov boundary of $A$). Hence $A|_{N}$ may be regarded as a uniform algebra on $\partial N$. Also, $z \in f(N) \setminus f(\partial N)$. By Lemma 2, $f(N)$ contains the component of $z$ in $C \setminus f(\partial N)$, so $f(N)$ is a neighborhood of $z$ as claimed.

**Proof of Theorem.** We will prove the following.

**Assertion.** If $f^{-1}(W)$ is nonempty, $\exists \ p \in f^{-1}(W)$ with a neighborhood in $M$ which is an analytic disk through $p$.

Assume for the moment that the assertion has been established. Given any $x \in f^{-1}(W)$ and any neighborhood $\emptyset$ of $x$, we may apply Lemma 3 to find a compact neighborhood $N$ of $x$ with properties (a), (b) and (c). Then the assertion may be applied to $f|_{N} \in A|_{N}$, to yield a point $p \in \emptyset$ with a neighborhood which is an analytic disk through $p$. Thus the set of such points is dense in $f^{-1}(W)$, and it is obviously open, so the Theorem follows.

**Proof of Assertion.** Suppose, on the contrary, that no point of $f^{-1}(W)$ has a neighborhood which is an analytic disk. We will use this assumption to find a point $z \in W$ such that $f^{-1}(z)$ is uncountable, contradicting the hypothesis of the Theorem. The uncountable set will consist of limit points obtained from sequences $\{x_{n_1n_2\cdots n_k}\}^{\infty}_{n=1}$ in $M$, which we will now define inductively.
Step 1. Since $f^{-1}(W)$ is nonempty, by Lemma 2 we have $f(f^{-1}(W)) = W$, so by Lemma 1 and the assumption that no point of $f^{-1}(W)$ has a neighborhood which is an analytic disk, there are distinct points $x_0, x_1 \in M$ and a corresponding $z_1 \in W$ such that $f(x_0) = f(x_1) = z_1$. By Lemma 3 there are disjoint compact $A$-convex neighborhoods $N_{i_1}$ of $x_{i_1}$ with the properties $N_{i_1} \subseteq f^{-1}(W)$, $z_1 \notin f(\partial N_{i_1})$, and $f(N_{i_1})$ is a neighborhood of $z_1$, $i_1 = 0, 1$. Choose $\varepsilon_1$ with $0 < \varepsilon_1 < 1$ so that

$$\{ z - z_1 \leq \varepsilon_1 \} \subseteq f(N_{i_1}) \cap f(\partial N_{i_1}), \quad i_1 = 0, 1. \quad (A_1)$$

Fix $i_1$ and look at

$$M_{i_1} = \{ y \in N_{i_1} \mid |f(y) - z_1| \leq \varepsilon_1 \} = N_{i_1} \cap f^{-1}(\{ z - z_1 \leq \varepsilon_1 \}).$$

$M_{i_1}$ is a nonempty compact $A$-convex set, so the maximal ideal space of $A|_{M_{i_1}}$ is $M_{i_1}$. Let $X_{i_1} = \{ y \in N_{i_1} \mid |f(y) - z_1| = \varepsilon_1 \}$. Observe that $\partial M_{i_1} \subseteq (\partial N_{i_1} \cup X_{i_1}) \cap M_{i_1} = X_{i_1}$ since $\partial N_{i_1} \cap M_{i_1} = \emptyset$ by $(A_1)$. Also observe that $M_{i_1}$ does not meet the Shilov boundary of $A$. By the local maximum modulus principle, the Shilov boundary of $A|_{M_{i_1}}$ is contained in $X_{i_1}$. Thus we may apply Lemma 1 with $X = X_{i_1}$, $A = A|_{X_{i_1}}$, $M = M_{i_1}$, $f = (f - z_1)/\varepsilon_1$, to conclude that (since there is no analytic disk through any point of $f^{-1}(W)$):

$$\text{linear measure } \{ z \in C \mid |z - z_1| = \varepsilon_1 \} \text{ is unique in } N_{i_1},$$

and $f^{-1}(z)$ is unique in $N_{i_1}$, $i_1 = 0, 1$. (Note that $f^{-1}(z) \cap N_{i_1} = f^{-1}(z) \cap M_{i_1}$ if $|z - z_1| = \varepsilon_1$. The set described in statement $(B_1)$ is measurable since Lemma 3 implies that it is closed. In fact, suppose that $\zeta_1, \zeta_2, \ldots \in C$, $\zeta_j \rightarrow \zeta \in C$, $|\zeta_j - z_1| = \varepsilon_1$ and there is precisely one $z_j \in N_{i_1}$ such that $f(z_j) = \zeta_j, j = 1, 2, \ldots$. Let $x \in f^{-1}(\zeta) \cap N_{i_1}$ and let $V$ be any neighborhood of $x$ in $M$. By Lemma 3, $f(V \cap N_{i_1})$ is a neighborhood of $\zeta$ in $C$, hence $\zeta_j$ is in $f(V \cap N_{i_1})$ for all $j \geq j_0$ for some $j_0$. But $f^{-1}(\zeta_j) \cap N_{i_1} = \{ z_j \}$, so $x_j$ is in $V$ for $j \geq j_0$. Thus $x_j$ converges to $x$, and $x$ is unique.)

This completes the first step.

Inductive hypothesis. Suppose that $z_n \in W$, $x_n \in M$, $N_n \subseteq M$ (where $a = i_1 \cdots i_n$) and $\varepsilon_n > 0$ have been chosen for $i_j = 0, 1$, $1 \leq j \leq n$. Assume that they satisfy the conditions: the $2^n$ points $x_n$ are all distinct; $f(x_n) = z_n$; $N_n$ is a compact $A$-convex neighborhood of $x_n$; $N_a \cap N_{b^*} = \emptyset$ (where $b$ = $j_1 \cdots j_n$) unless $i_1 = j_1, \ldots, i_n = j_n$; $N_n \subseteq \text{interior } N_{i_1\cdots i_n-1}$, if $n > 1; \varepsilon_n < 1/n^2$.

$(A_n)$

$$\{ z - z_n \leq \varepsilon_n \} \subseteq f(N_n) \cap f(\partial N_n);$$

$(B_n)$

$$\text{linear measure } \{ z \in C \mid |z - z_n| = \varepsilon_n \} \text{ is unique in } N_n,$$

and $f^{-1}(z)$ is unique in $N_n$.
Step $n+1$. By (B$_n$), $\exists z_{n+1} \in \{ |z - z_n| = \varepsilon_n \}$ such that for all $i_1 \cdots i_n$, there are two distinct points $x_c \in N_a$ (where $c = i_1 \cdots i_{n+1}$, $i_{n+1} = 0, 1$) with $f(x_c) = z_{n+1}$. By (A$_n$), we have $x_c \in N_a \setminus f^{-1}(f(\partial N_a))$. Fix $a = i_1 \cdots i_n$ and choose disjoint open sets $\emptyset_0, \emptyset_1$ with

$$x_c \in \emptyset_{i_{n+1}} \subseteq N_a \setminus f^{-1}(f(\partial N_a)), \quad i_{n+1} = 0, 1.$$  

Fix $i_{n+1} = 0$ or $1$ and apply Lemma 3 with

$$A = \overline{\mathcal{A}|_{2N_a}}, \quad X = \partial N_a, \quad M = N_a, \quad f = f|_{N_a}, \quad W = \text{component of } z_{n+1} \text{ in } C \setminus f(\partial N_a), \quad z = z_{n+1}, \quad \emptyset = \emptyset_{i_{n+1}}, \quad x = x_c.$$  

We conclude that there exist disjoint compact $A$-convex neighborhoods $N_c$ of $x_c$, $i_{n+1} = 0, 1$, with the properties $z_{n+1} \notin f(\partial N_c)$, $f(N_c)$ is a neighborhood of $z_{n+1}$. Do this for all indices $c = i_1 \cdots i_{n+1}$, $i_j = 0, 1$, $1 \leq j \leq n+1$.

From the definition of the $N_c$ it follows that we can choose an $\varepsilon_{n+1}$ with $0 < \varepsilon_{n+1} < 1/(n+1)^2$ so that for all $i_1 \cdots i_{n+1}$ we have

$$(A_{n+1}) \quad \{|z - z_{n+1}| \leq \varepsilon_{n+1}\} \subseteq f(N_c) \setminus f(\partial N_c).$$

Fix $c = i_1 \cdots i_{n+1}$ and define

$$M_c = \{y \in N_c \mid |f(y) - z_{n+1}| \leq \varepsilon_{n+1}\},$$

$$X_c = \{y \in N_c \mid |f(y) - z_{n+1}| = \varepsilon_{n+1}\}.$$  

As in Step 1, the maximal ideal space $\mathcal{A}|_M$ is $M_c$ and the Shilov boundary of $\mathcal{A}|_M$ is contained in $X_c$. ($(A_{n+1})$ guarantees that $\partial M_c \subseteq X_c$.) We may therefore apply Lemma 1 with

$$X = X_c, \quad A = \mathcal{A}|_{X_c}, \quad M = M_c, \quad f = (f - z_{n+1})/\varepsilon_{n+1}.$$  

We conclude that

$$(B_{n+1}) \quad \text{linear measure } \{z \in C \mid |z - z_{n+1}| = \varepsilon_{n+1}\} \text{ is unique in } N_c = 0.$$  

This equation holds for all indices $c = i_1 \cdots i_{n+1}$, $i_j = 0, 1$, $1 \leq j \leq n+1$, and the induction is complete.

By the above construction $\{z_n\}$ is Cauchy, so $z_n \to z$ for some $z \in C$; $z \in W$ since, for each $n$, $z_n \in f(N_a) \subseteq f(N_0) \cup f(N_1)$, a compact subset of $W$.

Let $I = (i_1, i_2, \cdots, i_n, \cdots)$ be an infinite sequence of 0’s and 1’s. Some subnet of the sequence $x_{i_1}, x_{i_1i_2}, x_{i_1i_2i_3}, \cdots$ converges to a point $x_f \in M$, and by continuity $f(x_f) = z$. In this way we associate an $x_f$ with each $I$. 

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Since there are uncountably many distinct $I$'s, the proof will be complete if we show that $x_I \neq x_J$ whenever $I \neq J$. Suppose therefore that $i_1 = j_1$, $i_2 = j_2$, \ldots, $i_m = j_m$, $i_{m+1} \neq j_{m+1}$. We know that $x_{i_1} \in N_{i_1}$, $x_{i_1 i_2} \in N_{i_1 i_2}$, \ldots and $N_{i_1} \supseteq N_{i_1 i_2} \supseteq N_{i_1 i_2 i_3} \cdots$ by the inductive hypotheses. Thus $x_I \in \bigcap_{n=1}^\infty N_n$. Similarly $x_J \in \bigcap_{n=1}^\infty N_n$ where $b = j_1 \cdots j_n$. But $N_{i_1} \cap N_{i_1} \cdots \cap N_{i_1 i_2 i_3} = \emptyset$ since $i_{m+1} \neq j_{m+1}$, so $x_I \neq x_J$.

B. Cole has pointed out that if one merely assumes that $f^{-1}(\lambda)$ is countable for $\lambda$ in a subset of $W$ of positive plane measure, the same proof shows that there is at least one point $p \in f^{-1}(W)$ with a neighborhood in $M$ which is an analytic disk through $p$.

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References

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