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THE EXISTENCE OF A SOLUTION OF $f(x) = kx$ FOR A CONTINUOUS NOT NECESSARILY LINEAR OPERATOR

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ABSTRACT. It is proved that if f is a continuous nonlinear operator on a Banach space E then $f(x)=kx$ has a solution when $|k|$ is sufficiently large.

Introduction. Let E be a Banach space and f a continuous operator on E. That is, f is a continuous, not necessarily linear mapping from E into E. In this note we prove that $f(x)=kx$ has a solution when $|k|$ is sufficiently large.

THEOREM. *Let f be a continuous operator on a real Banach space E. Then, for each $\delta>0$, there exists $k_\delta>0$ such that whenever $|k|\geqq k_\delta$, there exists $x \in A_\delta$ satisfying the equation $f(x)=kx$, where $A_\delta=\{x: \|x\|<\delta\}$.*

To prove the theorem we use the following fixed point theorem due to Altman [**1**, p. 97].

ALTMAN FIXED POINT THEOREM. *Let A be a convex open bounded set in a Banach space E containing* 0 *and let $\varphi: \bar{A} \to E$ be a continuous map satisfying $\|x-\varphi(x)\|^2 \geqq \|\varphi(x)\|^2 - \|x\|^2$ for all $x \in \partial A$. Then φ has a fixed point.*

PROOF OF THE THEOREM. Because f is a continuous operator on E, f is continuous at 0. Hence there exists $\delta_0>0$ such that $\|f(x)\|<1+|f(0)|$ for $\|x\|\leqq\delta_0$. Let $\varepsilon=\mathrm{Min}(\delta, \delta_0)$. Let $A=A_\varepsilon$ where $A_\varepsilon=\{x: \|x\|<\varepsilon\}$. Choose $k_\delta=\varepsilon^{-1}(1+|f(0)|)$. If $|k|\geqq k_\delta$, then $|k|\geqq\|x\|^{-1}f(x)$, for all $x \in \partial A$. Therefore $\|k^{-1}f(x)\|^2-\|x\|^2\leqq(\|x\|-\|k^{-1}f(x)\|)^2$, for all $x \in \partial A$. But, by the triangle inequality, $(\|x\|-\|k^{-1}f(x)\|)^2\leqq\|x-k^{-1}f(x)\|^2$. Therefore, whenever $k\geqq k_\delta$ we have $\|k^{-1}f(x)\|^2-\|x\|^2\leqq\|x-k^{-1}f(x)\|^2$ for all $x \in \partial A$. Hence $k^{-1}f(x)$ satisfies the conditions of Altman's theorem. Therefore for $k\geqq k_\delta$, there exists $x \in A$ such that $f(x)=kx$. Since $A\subset A_\delta$ the theorem follows.

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The following example shows that the conclusion of the theorem does not hold in an arbitrary normed linear space. Let S be the set of all sequences $\{a_n\}$ of real numbers such that $a_n=0$ for all but a finite number of n's. Now S is a normed linear space under the norm $\|\{a_n\}\|=\operatorname{Sup}|a_n|$. If we now define $f(\{a_n\})=(1, a_1, a_2, \cdots)$, it is obvious that f is continuous and that there does not exist $x \in S$ and k real, such that $f(x)=kx$.

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References

1. J. T. Schwartz, *Non-linear functional analysis*, Gordon and Breach, New York, 1969.

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