SELF-UNIVERSAL CRUMPLED CUBES AND
A DOGBONE SPACE

E. H. ANDERSON

Abstract. The question of whether each self-universal crumpled cube is universal is answered negatively by presenting an example of a dogbone space which is not topologically $E^3$ but which can be expressed as a sewing of two crumpled cubes, one of which is self-universal.

C. D. Bass and R. J. Daverman [2] presented a brief paper indicating that the solid Alexander horned sphere is an example of a crumpled cube which is self-universal but not universal, thus answering negatively the question asked by C. E. Burgess and J. W. Cannon in [4] of whether each self-universal crumpled cube is universal. The validity of the example presented by Bass and Daverman depends on a claim that a certain upper semicontinuous decomposition of $S^3$ into points and tame arcs, described in [2], is not topologically $S^3$, which in turn depends on the validity of four lemmas which are stated in [2, §2]. The proofs of these four lemmas appear to entail nontrivial arguments which are not included in [2].

In this note, we present an example of a dogbone space, an upper semicontinuous decomposition of $S^3$ into points and tame arcs whose nondegenerate elements can be expressed as the intersection of a tower of solid double tori, which is not topologically $S^3$. The dogbone space can be described as the result of a sewing of two crumpled cubes, one of which is self-universal. Thus, the question asked by Burgess and Cannon in [4] is answered negatively. The argument will be based essentially upon work by Casler [5] and the author [1].

Some recent work by Eaton [6] includes a different proof that the solid Alexander horned sphere $H$, used in the example presented by Bass and Daverman, is not universal. This was done by sewing $H$ to the crumpled cube $T$ described by Stallings [7] so that the wild points of $\text{Bd } H$ are sewn to the Cantor set of nonpiercing points of $T$. Other methods developed by Eaton, in papers cited in [6], should offer alternative ways

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to show that neither the decomposition space we describe here nor the one described by Bass and Daverman \[2\] is topologically $S^3$.

Definitions relating to crumpled cubes will be those of \[2\].

1. **The example.** The description of the dogbone space of this paper is modeled after that by Bing in \[3\]. As in Figure 1, let $H_1$ and $H_2$ denote the upper and lower half-spaces of $S^3$, and $P$ the $xy$-plane. Let $A_0$ denote a solid double torus which intersects $P$ in a disk. Then, embed solid double tori $A_1, \ldots, A_4$, linked as indicated, in $A_0$; although each of $A_1, \ldots, A_4$ is shown as a finite graph, it is topologically equivalent to $A_0$. Then, for each $i=1, \ldots, 4$, solid double tori $A_{i,1}, \ldots, A_{i,4}$ are embedded in $A_i$ such that there is a homeomorphism of $S^3$ onto itself which is the identity on the complement of some open set containing $A_0$ and takes $A_0$ onto $A_i$. Succeeding steps of the construction are to be described inductively.

Let $M=A_0 \cap \sum A_1 \cap \sum A_{i,j} \cap \sum A_{i,j,k} \cdots$. Let $G$ be the set whose elements are components of $M$ and one-point subsets of $S^3 - M$. Then, $G$ is an upper semicontinuous decomposition of $S^3$ into points and tame
arcs. Let $S^3/G$ denote the associated decomposition space, the dogbone space of this note.

From [1], we have that $S^3/G$ is not topologically $S^3$.

The methods of Casier [5] can be easily modified to show that $H^*_2$, the natural projection of $H_2$ in $S^3/G$, is a self-universal crumpled cube.

Since $H^*_1$, the natural projection of $H_1$ in $S^3/G$, is a crumpled cube and $S^3/G$ is the result of a sewing of $H^*_1$ and $H^*_2$, we have that $H^*_2$ is not a universal crumpled cube.

**Bibliography**

3. R. H. Bing, *A decomposition of $E^3$ into points and tame arcs such that the decomposition space is topologically different from $E^3$*, Ann. of Math. (2) 65 (1957), 484–500. MR 19, 1187.

**Department of Mathematics, Mississippi State University, State College, Mississippi 39762**