NONRECURSIVE RELATIONS AMONG THE ISOLS

ERIK ELLENTUCK¹

Abstract. The universal isol metatheorem is extended so as to deal with nonrecursive relations and countable Boolean operations.

1. Introduction. Let $\mathcal{L}$ be a first order language with equality containing an infinite list of individual variables $v_0, v_1, \cdots$, and for each $n<\omega$ an individual $n$, an $n$-ary function symbol $f$ for each almost recursive combinatorial $f: X^n \omega \rightarrow \omega$, and an $n$-ary relation symbol $R$ for each relation $R \subseteq X^n \omega$. Terms are built up from variables, constants, and function symbols by composition, and atomic formulas are of the form $\tau_0 = \tau_1$ or $R(\tau_0, \cdots, \tau_{n-1})$ where $\tau_0, \cdots, \tau_{n-1}$ are terms and $R$ is an $n$-ary relation symbol. Formulas and sentences are defined as usual. $\mathcal{L}$ has the standard interpretation in $\omega$ (write $\omega \models \mathfrak{A}[x]$ for "the assignment $x$ satisfies $\mathfrak{A}$ in $\omega$") and is interpreted in $\Lambda$ by letting $f$ and $R$ denote $f_\Lambda$ and $R_\Lambda$ respectively (write $\Lambda \models \mathfrak{A}[x]$ for "the assignment $x$ satisfies $\mathfrak{A}$ in $\Lambda$".

Throughout this paper $\mathfrak{A}$ will denote a quantifier-free conjunctive normal form formula all of whose free variables are among $v_0, \cdots, v_{k-1}$ and all of whose relation symbols occurring negated in some conjunct of $\mathfrak{A}$ are among $R_0, \cdots, R_{n-1}$. Whenever we wish to stress these symbols we write $\mathfrak{A}(R_0, \cdots, R_{n-1})$ and let $\mathfrak{A}(R'_0, \cdots, R'_{n-1})$ be the result of replacing each negated occurrence of $R_i$ in $\mathfrak{A}$ by $R'_i$ for $i < n$ (provided each $R_i$ and $R'_i$ have the same arity). We assume that the reader is familiar with the notions $\Lambda$, $\Lambda^\omega$, totally unbounded, specification ($S_\Lambda$), and Horn reduction. A set $R \subseteq X^k \omega$ is eventual if its complement $X^k \omega - R$ is not totally unbounded. It will also be convenient to introduce an improper notion $\Lambda^\omega \models (\forall v_0, \cdots, v_{k-1})\mathfrak{A}$ to mean $\Lambda \models \mathfrak{A}[x]$ for every $x \in X^k \Lambda^\omega$. Our starting point is the fundamental metatheorem of [3] which characterizes universal sentences in $\Lambda$.

Theorem 11.1 of [3]. If all relation symbols occurring in $\mathfrak{A}$ denote recursive relations then

(i) $\Lambda^\omega \models (\forall v_0, \cdots, v_{k-1})\mathfrak{A}$ if and only if there is a Horn reduction $\mathfrak{W}$ of $\mathfrak{A}$ such that $\{x \in X^k \omega : x \models \mathfrak{W}[x]\}$ is eventual.

¹ Prepared while the author was partially supported by National Science Foundation grant GP-11509.

© American Mathematical Society 1972
ERIK ELLENTUCK

(ii) \( \Lambda \models (\forall v_0, \ldots, v_{k-1}) \mathcal{A} \) if and only if \( \omega \models (\forall v_0, \ldots, v_{k-1}) \mathcal{A} \), and for each \( \sigma \subseteq k \) and \( h: \sigma \rightarrow \omega \) there is a Horn reduction \( \mathcal{A}' \) of \( \mathcal{A} \) such that

\[
S_h \{ x \in \mathcal{X}^k : \omega \models \mathcal{A}'[x] \} \text{ is eventual.}
\]

In our first result we remove the restriction that the relation symbols in \( \mathcal{A} \) denote recursive relations. Thus we have

**Theorem 1.** (i) \( \Lambda \models (\forall v_0, \ldots, v_{k-1}) \mathcal{A}(R_0, \ldots, R_{n-1}) \) if and only if for each sequence of recursively enumerable relations \( R'_0, \ldots, R'_{n-1} \) with \( R'_i \subseteq R_i \) for \( i < n \), there is a Horn reduction \( \mathcal{A}' \) of \( \mathcal{A} \) such that

\[
\{ x \in \mathcal{X}^k : \omega \models \mathcal{A}'(R'_0, \ldots, R'_{n-1})[x] \} \text{ is eventual.}
\]

(ii) \( \Lambda \models (\forall v_0, \ldots, v_{k-1}) \mathcal{A}(R_0, \ldots, R_{n-1}) \) if and only if

\[
\omega \models (\forall v_0, \ldots, v_{k-1}) \mathcal{A}(R_0, \ldots, R_{n-1})
\]

and for each sequence of recursively enumerable relations \( R'_0, \ldots, R'_{n-1} \) with \( R'_i \subseteq R_i \) for \( i < n \), and for each \( \sigma \subseteq k \) and \( h: \sigma \rightarrow \omega \) there is a Horn reduction \( \mathcal{A}' \) of \( \mathcal{A} \) such that \( S_h \{ x \in \mathcal{X}^k : \omega \models \mathcal{A}'(R'_0, \ldots, R'_{n-1})[x] \} \) is eventual. If all \( R_i \) are recursively enumerable suppress all mention of \( R_i \) above.

Note that the replacements made above of \( R'_i \) for \( R_i \) are only at the negated occurrences of \( R_i \) in \( \mathcal{A} \). Amusing consequences of Theorem 1 are:

(i) If \( R \subseteq \omega \) is immune with immune complement \( S = \omega - R \) then \( \Lambda = (\forall v_0)(R(v_0) \rightarrow S(v_0)) \) by Theorem 1(i) although nothing could be falser in \( \omega \). (ii) If \( R \subseteq \omega \) is immune and is expressed as the union \( R = S_0 \cup S_1 \) of two infinite disjoint sets then \( \Lambda = (\forall v_0)(R(v_0) \rightarrow S_0(v_0) \lor S_1(v_0)) \) by Theorem 1 (ii) even though neither Horn reduction is eventual in \( \omega \). Note, however, that these examples readily follow from the fact that \( R_\Lambda = R \) for immune \( R \) (cf. Theorem 4.1 of [3]). These examples in no way illustrate the strength of Theorem 1 which is concerned rather with getting results when function symbols are present. It would be desirable of course to develop a theory which would allow for not necessarily recursive almost combinatorial functions; however, the well-known result that in general composition of such functions does not commute with their extension makes us pessimistic of such a possibility.

Our second result concerns a generalization of Theorem 1 to a class of infinitary universal sentences in \( \mathcal{L}_{\omega_1\omega_1} \). The basic symbols of this language will be the same as those of \( \mathcal{L} \) except now we allow countable conjunctions, disjunctions, and application of countable homogeneous quantifier blocks. In this paper \( \mathcal{B} \) will denote a quantifier free formula of \( \mathcal{L}_{\omega_1\omega_1} \) consisting of a countable conjunction of countable disjunctions of atomic formulae and their negations. We assume that the free variables of \( \mathcal{B} \) are among \( v_0, v_1, \ldots \), and that the relation symbols occurring negated
in some conjunct of $\mathcal{B}$ are among $R_0, R_1, \cdots$. We stress these symbols by writing $\mathcal{B}(R_0, R_1, \cdots)$ and let $\mathcal{B}(R_0', R_1', \cdots)$ be the result of replacing each negated occurrence of $R_i$ in $\mathcal{B}$ by $R_i'$ for $i<\omega$ (provided each $R_i$ and $R_i'$ have the same ary-ness). Let $v=\langle v_0, v_1, \cdots \rangle$ and express the universal closure of $\mathcal{B}$ as $(\forall v)\mathcal{B}$. Let $\mathcal{B}'$ be any conjunct of $\mathcal{B}$. We say that $\mathcal{B}''$ is a finite approximation of $\mathcal{B}'$ if $\mathcal{B}''$ is obtained from $\mathcal{B}'$ by striking out all but a finite positive number of disjuncts of $\mathcal{B}'$. Thus $\mathcal{B}''$ is an ordinary formula of $\mathcal{L}$ whose universal quantification can be expressed as $(\forall v)\mathcal{B}''$.

**Theorem 2.** $\Lambda_n \models (\forall v)\mathcal{B}(R_0, R_1, \cdots)$ if and only if for each sequence of recursively enumerable relations $R_0', R_1', \cdots$, with $R_i' \subseteq R_i$ for $i<\omega$, and for each conjunct $\mathcal{B}'$ of $\mathcal{B}$, there is a finite approximation $\mathcal{B}''$ of $\mathcal{B}'$ such that $\Lambda_n \models (\forall v)\mathcal{B}''(R_0', R_1', \cdots)$. If all $R_i$ are recursively enumerable suppress all mention of $R_i'$ above.

Amusing consequences of Theorem 2 are: (i) If $R_i = \omega - \{i\}$ and $\mathcal{B}$ is $(R_1(v_0), R_2(v_0), \cdots, v_0=0)$ then $(\forall v)\mathcal{B}$ is an infinitary universal Horn sentence true in $\omega$ but false in $\Lambda$ since no finite approximation of $\mathcal{B}$ holds eventually in $\omega$. (ii) If $R \subseteq \omega$ is a recursively enumerable nonrecursive set and $S_0, S_1, \cdots$ is an enumeration of all the recursive subsets of $R$, and $\mathcal{B}$ is $R(v_0) \rightarrow (S_0(v_0) \vee S_1(v_0) \vee \cdots)$, then $(\forall v)\mathcal{B}$ is true in $\omega$ but false in $\Lambda$ since no finite approximation of $\mathcal{B}$ holds eventually in $\omega$. Example (ii) is Corollary 3.9 of [4]. It is an interesting result because it implies that the method of extending relations by frames is indeed stronger than the method of extending relations by equations.

2. **Proofs.** The proof of Theorem 1 depends on a simple lemma which was probably overlooked because no one was looking for it. We have

**Lemma 1.** If $R \subseteq X^{<\omega}$, $S \subseteq X^{<\omega}$, and $f_i : X^{<\omega} \rightarrow \omega$, for $i<n$, are recursive combinatorial functions such that $(\forall x \in X^{<\omega}) \,(x \in S \iff \langle f_0(x), \cdots, f_{n-1}(x) \rangle \in R)$ then for all $x \in X^{<\Lambda}$, $x \in S_\Lambda$ if and only if $\langle f_0(\Lambda(x)), \cdots, f_{(n-1)}(\Lambda(x)) \rangle \in R_\Lambda$.

**Proof.** We extend ordinary set theoretic notation componentwise to $k$-tuples as follows. Let $\mathcal{P}$ be the power set operation. If $\alpha, \beta \in X^k\mathcal{P}(\omega)$ let $\alpha \subseteq \beta$ if $\alpha_i \subseteq \beta_i$ for $i<k$, let $\alpha \cap \beta$ be that element of $X^k\mathcal{P}(\omega)$ such that $(\alpha \cap \beta)_i = \alpha_i \cap \beta_i$ for $i<k$ and let $|\alpha|$ be that sequence of cardinals such that $|\alpha_i| = |x_i|$ for $i<k$. If $A \subseteq X^k\mathcal{P}(\omega)$ let $\bigcup A \in X^k\mathcal{P}(\omega)$ satisfy $(\bigcup A)_i = \bigcup \{x_i : x \in A \}$ for $i<k$.

Finally let $Q$ be the set of all finite subsets of $\omega$. Now for our proof. Let $x=\langle x_0, \cdots, x_{k-1} \rangle \in S_\Lambda$ and choose $\xi=\langle \xi_0, \cdots, \xi_{k-1} \rangle$ such that $x_i = \text{Req}(\xi_i)$ for $i<k$. Let $G$ be a recursive...
S-frame such that $\xi$ is attainable from $G$, in symbols $\xi \in \mathcal{A}(G)$ and let $\varphi_i, i < n,$ be recursive combinatorial operators inducing $f_i.$ Write $\varphi(\alpha)$ for $(\varphi_0(\alpha), \ldots, \varphi_{n-1}(\alpha)).$ One direction of our lemma follows by showing that $F = \{ \varphi(\alpha) : \alpha \in G \}$ is a recursive R-frame for which $\varphi(\xi) \in \mathcal{A}(F).$ Now $F^*$ is recursively enumerable because $F^* = \{ \beta : (\exists \alpha \in G^*)(\beta \subseteq \varphi(\alpha)) \}.$ If $\gamma \in F^*$ then $\gamma \subseteq \varphi(\alpha)$ for some $\alpha \in G.$ Next we define

$$\Sigma(\gamma) = \varphi(C_0(\bigcup \{ y : (\exists i < n)(\exists x \in \gamma_i) (y = \varphi_i^{-1}(x)) \}))$$

and note that $\Sigma$ is partial recursive and $\gamma \subseteq \Sigma(\gamma) \subseteq \varphi(\alpha).$ Since $\Sigma(\gamma) \in F,$ it is clearly the required $C_\beta.$ $F$ is closed under $\bigcap$ because $G$ is closed under $\bigcap$ and $\varphi$ is multiplicative. $\beta \in F$ implies $|\beta| \in R$ follows by hypothesis and the definition of $F.$ Thus $F$ is a recursive R-frame. If $\gamma \in \mathcal{X}^nQ$ and $\gamma(\xi)$ then $\alpha = \bigcup \{ y : (\exists i < n)(\exists x \in \gamma_i) (y = \varphi_i^{-1}(x)) \} \subseteq \xi$ and hence $\gamma \subseteq \varphi(\alpha) \subseteq \varphi(C_0(\alpha)) \subseteq \varphi(\xi).$ But $C_\beta(\gamma) = \varphi(C_0(\alpha)) \in F$ so that $\varphi(\xi) \in \mathcal{A}(F).$ Conversely suppose that $F$ is a recursive R-frame such that $\varphi(\xi) \in \mathcal{A}(F).$ The other direction of our lemma will follow by showing that $G = \{ \alpha : \varphi(\alpha) \in F \}$ is a recursive S-frame for which $\xi \in \mathcal{A}(G).$ Let us first note that $H = \{ \varphi(\alpha) : \alpha \in X^kQ \}$ is a recursive frame and that $\varphi(\xi) \in \mathcal{A}(H).$ Since $\varphi(\xi) \in \mathcal{A}(F)$ and $\varphi(\xi)$ is isolated, 3.6 of [3] implies that $F \cap H$ is a recursive R-frame and $\varphi(\xi) \in \mathcal{A}(F \cap H).$ Hence there is no loss of generality by assuming that $F \subseteq H.$ Then

(1) \[ \alpha = \varphi(\bigcup \{ y : (\exists i < n)(\exists x \in \alpha_i) (y = \varphi_i^{-1}(x)) \}) \]

for $\alpha \in F$ follows from elementary properties of combinatorial operators. We show that $G^*$ is recursively enumerable, that $G$ is closed under $\bigcap,$ and that $\alpha \in G$ implies $|\alpha| \in S$ in exactly the same way as before. If $\gamma \in G^*$ then $\gamma \subseteq \alpha$ for some $\alpha \in G.$ Now define

$$\Sigma(\gamma) = \gamma \cup \bigcup \{ y : (\exists i < n)(\exists x \in C_\beta(\varphi(\gamma))) (y = \varphi_i^{-1}(x)) \}$$

and note that $\Sigma$ is partial recursive and $\gamma \subseteq \Sigma(\gamma) \subseteq \alpha.$ Moreover, from (1) we see that $\varphi(\Sigma(\gamma)) = C_\beta(\varphi(\gamma)) \in F$ and consequently $\Sigma(\gamma)$ is the required $C_\beta.$ Thus $G$ is a recursive S-frame. If $\gamma \in \mathcal{X}^nQ$ and $\gamma \subseteq \xi$ then $\varphi(\gamma) \subseteq C_\beta(\varphi(\gamma)) \subseteq \varphi(\xi).$ It immediately follows from the definition of $\Sigma(\gamma)$ that $\gamma \subseteq \Sigma(\gamma) \subseteq \xi,$ i.e., $\xi \in \mathcal{A}(F).$ Q.E.D.

**Lemma 2.** The same as Lemma 1 except that we replace the $f_i, i < n,$ by almost recursive combinatorial functions.

**Proof.** For notational ease assume that $R, S \subseteq \omega$ and $f$ is unary. Our proof below works equally well in the general case. Let us suppose that $f$ is almost recursive combinatorial and $(\forall x \in \omega)(x \in S \iff f(x) \in R).$ Let $\bar{R} = \{ x \in X^\omega : x_0 - x_1 \in R \}$ and let $f^+, f^-$ be a pair of unary recursive
combinatorial functions such that $$(\forall x \in \omega)(f(x) = f^+(x) - f^-(x))$$. By definition of $\tilde{R}$, $$(\forall x \in \omega)(x \in S \iff \langle f^+(x), f^-(x) \rangle \in \tilde{R})$$ and hence by Lemma 1 we have $$(\forall x \in \Lambda)(x \in S_\Lambda \iff \langle f^+(x)_\Lambda, f^-(x)_\Lambda \rangle \in \tilde{R}_\Lambda)$$. The sentences

$$(\forall v_0, v_1, v_2)(v_0 + v_1 = v_2 \land \tilde{R}(v_2, v_0) \rightarrow R(v_1)),$$

$$(\forall v_0, v_1, v_2)(v_0 + v_1 = v_0 \land R(v_1) \rightarrow \tilde{R}(v_2, v_0))$$

are true in $\omega$, have no atomic formula containing both a function symbol and a relation symbol denoting a nonrecursive relation, and hence by Corollary 11.2 of [3], are true in $\Lambda$. Thus for all $x_0, x_1 \in \Lambda$ if $x_0 \geq x_1$ then $\langle x_0, x_1 \rangle \in \tilde{R}_\Lambda$ if and only if $x_0 - x_1 \in R_\Lambda$. Since $f^+(x)_\Lambda = f^+(x) - f^-(x)$ for all $x \in \Lambda$ our lemma follows immediately. Q.E.D.

We also need the following result which is proved in [2],

**Lemma 3.** If $R \subseteq X^{\omega}$, $S \subseteq \omega$ are recursively enumerable and $f_i: \omega \rightarrow \omega$ for $i < n$ are combinatorial functions (which are not necessarily recursive) such that $$(\forall x \in \omega)(x \in S \iff \langle f_0(x), \ldots, f_{n-1}(x) \rangle \in R)$$ then there is an immune set $\theta$ such that if $\xi \in \theta$ is infinite and $x = \text{Req}(\xi)$ then $x \in S_\Lambda$ if and only if $\langle f_0_{\Lambda}(x), \ldots, f_{n-1}_{\Lambda}(x) \rangle \in R_\Lambda$.

**Proof of Theorem 1.** (A sketch; refer to [3] for all the details.) As in the proof of Theorem 11.1 of [3] part (ii) reduces to part (i), and for (i) it suffices to consider the case where $\exists$ is a single conjunct. We may also dispense with equality and assume terms have been collapsed to single function symbols. Thus we may assume that $\exists$ has the form

$$(\exists) \bigwedge R_i(f_i(x)) \rightarrow \bigvee S_i(g_i(x))$$

where $\bigwedge$ denotes repeated conjunction (disjunction) and if $R$ say, is $j$-ary, we have written $R(f(v))$ as short for

$R(f_0(v_0, \ldots, v_{k-1}), \ldots, f_{j-1}(v_0, \ldots, v_{k-1}))$.

Now suppose that $\exists$ satisfies the arithmetical condition on $\omega$ given in (i) and that $x \in X^\omega$ satisfies $\Lambda \models \bigwedge_{i < n} R_i(f_i(x))[x]$, i.e., $f_{i\Lambda}(x)_\Lambda \in R_{i\Lambda}$ for $i < n$. By the definition of extension by frames this implies that there are recursively enumerable relations $R'_{i\Lambda}$, $R'_{i\Lambda} \subseteq R_{i\Lambda}$ for $i < n$, such that $f_{i\Lambda}(x)_\Lambda \in R'_{i\Lambda}$, i.e., $\Lambda \models \bigwedge_{i < n} R'_i(f_i(x))[x]$. Hence by hypothesis there is a Horn reduction $\exists'$ of $\exists$, such that $\exists'(R'_0, \ldots, R'_{n-1})$ has the form

$$(\exists) \bigwedge_{i < n} R'_i(f_i(x)) \rightarrow S(g(x)),$$

where $S(g(x))$ is one of $S_i(g_i(x))$, and which eventually holds in $\omega$. Let $R'_{\exists'} = \{ x \in X^{\omega}: f_i(x) \in R'_{i\Lambda}\}$, $S'' = \{ x \in X^{\omega}: g(x) \in S\}$ so that by making the
appropriate replacements in (3),

\[ \prod_{i < n} R'_i(v) \rightarrow S'(v) \]

(4) eventually holds in \( \omega \). By Corollary 11.2 of [3], (4) will also hold in \( \Lambda^\omega \). By Lemma 2, \( x \in R'_i^{\omega} \) for \( i < n \) and hence by (4), \( x \in S'^{\omega} \) so that again by Lemma 2 \( g_{\Lambda}(x) \in S_{\Lambda} \), i.e., \( \Lambda \models S(g(v))[x] \) and therefore \[ \Lambda \models \sum_{i < m} S_i(g_i(v))[x]. \]

Thus (2) holds in \( \Lambda^\omega \). Conversely suppose that the arithmetical condition on \( \omega \) in (i) does not hold, i.e., that there are recursively enumerable \( R'_i, R'_i \subseteq R_i \) for \( i < m \) such that for each \( j < m \)

\[ \prod_{i < n} R'_i(f_i(v)) \rightarrow S'_j g_j(v) \]

(5) does not eventually hold in \( \omega \). Define \( R'_i, S'_j \) as above so that

\[ \prod_{i < n} R'_i(v) \rightarrow S'_j(v) \]

(6) does not eventually hold in \( \omega \) for each \( j < m \), and note that the \( R'_i \) are recursively enumerable. Now by the method described in [1] we can construct a \( k \)-tuple \( h(x) = (h_0(x), \ldots, h_{k-1}(x)) \) of not necessarily recursive, but unary combinatorial functions such that for each \( i < n, j < m, \) and \( x < \omega, h(x) \in R'_i \), but \( h(x) \notin S'_j \) for infinitely many values of \( x \). By Lemma 3 there is an immune set \( \theta \subseteq \omega \) such that for every infinite subset \( \xi \subseteq \theta \), if \( x = \text{Req} (\xi) \) then \( h_{\Lambda}(x) \in R'_i^{\Lambda} \) for \( i < n \) (note that here is the place where we really use the fact that each \( R'_i \) is recursively enumerable). Further by the usual category argument on the Cantor space of subsets of \( \theta \) we can actually find a \( \xi_0 \subseteq \theta \) such that if \( x_0 = \text{Req} (\xi_0) \) then \( u_0 = h_{\Lambda}(x_0) \notin S'_j^{\Lambda} \) for \( j < m \). We can also guarantee that \( u_0 \in \times^k \Lambda^\omega \) by choosing the components of \( h \) to be strictly increasing functions. Now use Lemma 2 to show that \( f_{i \Lambda}(u_0) \in R'_i^{\Lambda} \subseteq R_i^{\Lambda} \) for \( i < n \) and \( g_{j \Lambda}(u_0) \notin S'_j^{\Lambda} \) for \( j < m \). Go back to our formalism and see that \( u_0 \) is the required counterexample for (2). Q.E.D.

Having seen the proof of Theorem 1, Theorem 2 requires little argumentation. The positive aspect of the theorem is self-evident; it was the negative side of the theorem which used a counterexample obtained from nonrecursive combinatorial functions which gave us difficulty. In Theorem 1 \( h \) was nonrecursive because the \( S_j \) were nonrecursive. In Theorem 2 \( h \) (an \( \omega \)-tuple of unary combinatorial functions) is nonrecursive for that reason and also because there need be no effective enumeration of the infinitely many \( S_j \). The details for the construction of \( h \) are tedious.
though straightforward in principle. We use Theorem 1 to show that certain sets are totally unbounded. The interested reader can see this argument in the proof of Theorem 2 of [2]. It should be noted that our negative results could also be gotten from a somewhat altered version of the compactness theorem of [4].

BIBLIOGRAPHY