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A THEOREM ON HOLOMORPHIC MAPPINGS INTO BANACH SPACES WITH BASIS

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ABSTRACT. Let F be a mapping from a complex Banach space into another complex Banach space with a Schauder basis, such that each coordinate composition mapping is holomorphic. Necessary and sufficient conditions are given that F be holomorphic.

1. **Introduction and definitions.** A mapping F from an open subset U of a complex Banach space X into a complex Banach space Y is said to be analytic if, for each $x_0 \in U$, there is a bounded linear mapping L_{x_0} ($\in \mathcal{L}(X, Y)$) from X to Y such that

$$\frac{\|F(x) - F(x_0) - L_{x_0}(x - x_0)\|}{\|x - x_0\|} \to 0 \quad \text{as } \|x - x_0\| \to 0.$$

We will consider Banach spaces Y with a Schauder basis. That is, there is a sequence $\{y_j\}$ in Y such that for every $y \in Y$, there exists a unique sequence of scalars $\{\alpha_j\}$ with $\|y - \sum_{j=1}^{J} \alpha_j y_j\| \to 0$ as $J \to \infty$. Hence if F maps U into Y, a Banach space with basis, we will write $F(x) = \sum_{j=1}^{\infty} g_j(x) y_j$.

A mapping F from U into Y is said to be locally bounded if, for each $x_0 \in U$, there exists a $\delta > 0$ such that $B_\delta(x_0) \subset U$ and

$$\sup\{\|F(x)\| : x \in B_\delta(x_0)\} < \infty.$$

Again assuming Y has the basis $\{y_j\}$, we say that the mapping $F: U \to Y$ is normal if, for each compact set $K \subset U$ and each $\varepsilon > 0$, there exists a J_0 such that $\|\sum_{j=J}^{\infty} g_j(x) y_j\| < \varepsilon$ for each $J \geqq J_0$ and all $x \in K$.

The space of continuous linear functionals on Y is written Y^*. In particular, the projection functionals $y = \sum_{j=1}^{\infty} \alpha_j y_j \in Y \to \alpha_j \in C$ will be written as e_j. A closed linear subspace $T \subset Y^*$ is a determining space for Y if the following equation is valid for all $y \in Y$:

$$\|y\| = \sup\{|t(y)| : t \in T, \|t\| \leqq 1\}.$$

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2. **The principal theorem.** As above, U is an open subset of a complex Banach space X and Y is a Banach space with basis $\{y_j\}$.

THEOREM 1. *Let $F: U \to Y$ and assume that, for each j, the complex valued functions $e_j \circ F(x)$ are analytic. Then, the following are equivalent:*

(a) *F is analytic.*

(b) *F is continuous.*

(c) *F is a normal mapping.*

(d) *F is weakly continuous.*

(e) *F is locally bounded.*

PROOF. (a) implies (b) is trivial. Assume that (b) holds but not (c). Then there is a compact set $K \subset U$ and an $\varepsilon_0 > 0$ such that, for every n, there is a $j(n) \geqq n$ and an $x_n \in K$ such that $\|\sum_{j=j(n)}^{\infty} g_j(x_n) y_j\| \geqq \varepsilon_0$. By replacing the sequence $\{x_n\}$ by a subsequence, we may assume that x_n converges to x^* which must be in K. By [3, p. 19], we can replace the norm $\| \ \|$ of Y with

$$(*) \qquad |||y||| = \sup_{1 \leqq J < \infty} \left\{ \left\| \sum_{j=1}^{J} \alpha_j y_j \right\| \right\},$$

where $y = \sum_{j=1}^{\infty} \alpha_j y_j$. The norm $||| \ |||$ is equivalent to $\| \ \|$, say $k_1 |||y||| \leqq \|y\| \leqq k_2 |||y|||$. For notational purposes, we will set $S_J(x) = \sum_{j=1}^{J} g_j(x) y_j$ and $E_J(x) = \sum_{j=J+1}^{\infty} g_j(x) y_j$, so that $F(x) = S_J(x) + E_J(x)$ for all J. For all x and x' in U,

$$\begin{aligned} \|E_J(x) - E_J(x')\| &= \|(F(x) - S_J(x)) - (F(x') - S_J(x'))\| \\ &\leqq \|F(x) - F(x')\| + k_2 |||S_J(x) - S_J(x')||| \\ &\leqq \|F(x) - F(x')\| + k_2 |||F(x) - F(x')||| \\ &\leqq k\|F(x) - F(x')\|, \quad \text{for some constant } k. \end{aligned}$$

Returning to the sequence $\{x_n\}$ in K, we obtain the inequality

$$\begin{aligned} k\|F(x_n) - F(x^*)\| &\geqq \|E_{j(n)}(x_n) - E_{j(n)}(x^*)\| \\ &\geqq \|E_{j(n)}(x_n)\| - \|E_{j(n)}(x^*)\| \geqq \varepsilon_0 - \|E_{j(n)}(x^*)\|. \end{aligned}$$

The continuity of F yields a contradiction, hence (b) implies (c).

To show (c) implies (d), we must prove that $\phi \circ F: U \to C$ is continuous for each $\phi \in Y^*$. If $x_n \to x$ in U, then since $\{x_n\} \cup \{x\}$ is compact, we have

$$\begin{aligned} |\phi \circ F(x) - \phi \circ F(x_n)| &= |\phi(S_J(x) + E_J(x)) - \phi(S_J(x_n) + E_J(x_n))| \\ &\leqq \|\phi\| \ \|S_J(x) - S_J(x_n)\| \\ &\quad + \|\phi\| \{\|E_J(x)\| + \|E_J(x_n)\|\}. \end{aligned}$$

The right-hand side can be made arbitrarily small for sufficiently large n, proving (c) implies (d).

To prove (d) implies (e), let $x \in U$ and assume that for some sequence $x_n \to x$, $\|F(x_n)\| \to \infty$. Consider the evaluation mappings $(F(x_n))\hat{} \in Y^{**}$ and use the weak continuity to see that

$$|(F(x_n))\hat{}(\phi)| \leqq M = M(\phi)$$

for all n. The uniform boundedness principle implies that $\|(F(x_n))\hat{}\| = \|F(x_n)\|$ cannot be arbitrarily large.

Finally, we show that (e) implies (a). Without loss of generality, we may assume that the norm of Y is given by (*). Let $\mathscr{S}$ be the linear subspace of Y^* spanned by the projection functionals $\{e_j\}$. If ϕ is any function in the closure of $\mathscr{S}$, we claim that $\phi \circ F: U \to C$ is analytic. For this we must show that if $x_0 \in U$, then $\phi \circ F$ is analytic near x_0. Assume that $\|F(x)\| \leqq M$ for all $x \in B_\delta(x_0) \subset U$. There is a sequence $\sum \alpha_j^{(n)} e_j = \phi_n \in \mathscr{S}$ such that $\|\phi - \phi_n\| \to 0$ as $n \to \infty$; that is, given $\varepsilon > 0$, there is N such that

$$\sup_{\|y\| \leqq 1} |\phi_n(y) - \phi(y)| < \varepsilon$$

for all $n \geqq N$. In our situation, this yields $|\phi_n(F(x)) - \phi(F(x))| < \varepsilon M$, if $\|x - x_0\| < \delta$. Each $\phi_n \circ F$ is analytic, and hence so too is $\phi \circ F$.

We will prove next that $\overline{\mathscr{S}}$ is determining for Y, which will imply (a). First, N. Dunford has shown that a function f, mapping a domain in the complex plane into a Banach space Y, is holomorphic if and only if $\phi \circ f$ is holomorphic for all ϕ in a determining manifold of Y^* [**1**, p. 354]. The required result then follows from the local boundedness of F [**2**, p. 112].

To prove that $\overline{\mathscr{S}}$ is determining, we let $y = \sum_{j=1}^{\infty} \alpha_j y_j \in Y$. Letting $\varepsilon > 0$ be arbitrary, there is an integer J such that $\|S_J\| > \|y\| - \varepsilon$. Consider the subspace of Y spanned by $\{y_1, \cdots, y_J\}$, Y_J, and the subspace of Y^* spanned by $\{e_1, \cdots, e_J\}$, Y_J^*. It is clear that by the Hahn Banach Theorem, we can choose $T \in Y_J^*$ such that $T(S_J) = \|S_J\|$, $\|T\| = 1$. Of course, we can consider T as a member of $\mathscr{S}$ and this implies that $\overline{\mathscr{S}}$ is determining. This completes the proof.

To illustrate the sharpness of Theorem 1, we give the following example. Let F be the mapping from the unit disc Δ of C into l^2 defined as follows. Let $z_n = \frac{1}{2} e^{i\theta n}$, where $\theta_n \to 0$, $\theta_n > 0$. Choose a wedge $B_n = \{re^{i\theta}: 0 < r < 1, |\theta - \theta_n| < \varepsilon_n\}$, where the ε_n are chosen so small that the B_n are disjoint. Let A_n be the complement of B_n in Δ. By the Runge approximation theorem, there are polynomials p_n which satisfy

$$|p_n(z)| \leqq \tfrac{1}{2} \quad \text{on } A_n, \qquad p_n(z_n) = 2.$$

Define $F(z) = (p_n^n(z))$. For all $z \in \Delta$, $F(z) \in l^2$. However, $\|F(z_n)\| \geqq 2^n$, and so F is not locally bounded at $\frac{1}{2}$. Hence, even though $e_j \circ F(z)$ is a polynomial for all j, F is not holomorphic.

3. **Some comments and a problem.** Theorem 1 was initially done for l^p spaces, with cases for $p=\infty$, $1<p<\infty$, and finally $p=1$. Let $F: U \to l^p$ $(1 \leqq p \leqq \infty)$ with $F(x)=(g_j(x))$ and each $g_j: U \to C$ holomorphic. If $x_0 \in U$ and $B_\delta(x_0) \subset U$, we can define the following set C_n for each positive integer n:

$$C_n = \{x: \|x - x_0\| < \delta \text{ and } \|F(x)\| \leqq n\}.$$

C_n is closed and $\bigcup_{n=1}^{\infty} C_n = B_\delta(x_0)$. By the Baire category theorem, some C_{n_0} has nonempty interior, and so, by Theorem 1, F is analytic on $\operatorname{int}(C_{n_0})$. Hence, there is an open dense subset of U on which F is analytic.

A result of Theorem 1(e) in the case where $X=C$ and $Y=l^p$ $(1 \leqq p < \infty)$ is that if $F: U \to l^p$ is holomorphic, then $F(z)$ is locally bounded. Therefore $\{g_j(z)\}$ is a normal family in the sense of Montel and $g_j(z) \to 0$ uniformly on compact subsets of U. We ask whether the normality of the sequence $\{g_j\}$ is sufficient to imply that $F: U \to l^p$ is holomorphic.

For $p=\infty$, the answer is affirmative. For, if $z_0 \in U$ and $|g_j(z)| \leqq M$ on $B_\delta(z_0)$, then $|g_j'(z_0)| \leqq M/\delta$. Hence, for all $z_0 \in U$, $\{g_j'(z_0)\} \in l^\infty$. The linear transformation $z \to (g_j'(z_0) \cdot z)$ is the Fréchet derivative of F at z_0. To see this, consider

$$\left| \frac{g_j(z) - g_j(z_0)}{z - z_0} - g_j'(z_0) \right| = \left| \sum_{n=2}^{\infty} \frac{g_j^{(n)}(z_0)}{n!} (z - z_0)^{n-1} \right|$$
$$\leqq |z - z_0| \sum_{n=2}^{\infty} \frac{M}{\delta^n} |z - z_0|^{n-2}.$$

The last sum is independent of j, which gives the result.

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