A THEOREM ON HOLOMORPHIC MAPPINGS INTO BANACH SPACES WITH BASIS

RICHARD ARON AND JOSEPH A. CIMA

Abstract. Let $F$ be a mapping from a complex Banach space into another complex Banach space with a Schauder basis, such that each coordinate composition mapping is holomorphic. Necessary and sufficient conditions are given that $F$ be holomorphic.

1. Introduction and definitions. A mapping $F$ from an open subset $U$ of a complex Banach space $X$ into a complex Banach space $Y$ is said to be analytic if, for each $x_0 \in U$, there is a bounded linear mapping $L_{x_0}$ ($\in \mathcal{L}(X, Y)$) from $X$ to $Y$ such that

$$\frac{\|F(x) - F(x_0) - L_{x_0}(x - x_0)\|}{\|x - x_0\|} \to 0 \quad \text{as} \quad \|x - x_0\| \to 0.$$ 

We will consider Banach spaces $Y$ with a Schauder basis. That is, there is a sequence $\{y_j\}$ in $Y$ such that for every $y \in Y$, there exists a unique sequence of scalars $\{\alpha_j\}$ with $\|y - \sum_{j=1}^{\infty} \alpha_j y_j\| \to 0$ as $J \to \infty$. Hence if $F$ maps $U$ into $Y$, a Banach space with basis, we will write $F(x) = \sum_{j=1}^{\infty} g_j(x)y_j$.

A mapping $F$ from $U$ into $Y$ is said to be locally bounded if, for each $x_0 \in U$, there exists a $\delta > 0$ such that $B_{\delta}(x_0) \subset U$ and

$$\sup\{\|F(x)\| : x \in B_{\delta}(x_0)\} < \infty.$$ 

Again assuming $Y$ has the basis $\{y_j\}$, we say that the mapping $F: U \to Y$ is normal if, for each compact set $K \subset U$ and each $\epsilon > 0$, there exists a $J_0$ such that $\|\sum_{j=1}^{\infty} g_j(x)y_j\| < \epsilon$ for each $J \geq J_0$ and all $x \in K$.

The space of continuous linear functionals on $Y$ is written $Y^*$. In particular, the projection functionals $y = \sum_{j=1}^{\infty} \alpha_j y_j \in Y \to x_j \in C$ will be written as $e_j$. A closed linear subspace $T \subset Y^*$ is a determining space for $Y$ if the following equation is valid for all $y \in Y$:

$$\|y\| = \sup\{|t(y)| : t \in T, \|t\| \leq 1\}.$$ 

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2. The principal theorem. As above, \( U \) is an open subset of a complex Banach space \( X \) and \( Y \) is a Banach space with basis \( \{ y_j \} \).

**Theorem 1.** Let \( F : U \to Y \) and assume that, for each \( j \), the complex valued functions \( e_j \circ F(x) \) are analytic. Then, the following are equivalent:

(a) \( F \) is analytic.

(b) \( F \) is continuous.

(c) \( F \) is a normal mapping.

(d) \( F \) is weakly continuous.

(e) \( F \) is locally bounded.

**Proof.** (a) implies (b) is trivial. Assume that (b) holds but not (c). Then there is a \( j(n) \geq n \) and an \( x_n \in K \) such that \( \| \sum_{j=j(n)}^\infty g_j(x_n) y_j \| \geq \epsilon_0 \). By replacing the sequence \( \{ x_n \} \) by a subsequence, we may assume that \( x_n \) converges to \( x^* \) which must be in \( K \). By [3, p. 19], we can replace the norm \( \| \cdot \| \) of \( Y \) with

\[
(*) \quad \| y \| = \sup_{1 \leq J < \infty} \left\{ \left\| \sum_{j=1}^J a_j y_j \right\| \right. ,
\]

where \( y = \sum_{j=1}^\infty a_j y_j \). The norm \( \| \cdot \| \) is equivalent to \( \| \cdot \| \) on \( y \), say \( k_1 \| y \| \leq \| y \| \leq k_2 \| y \| \). For notational purposes, we will set \( S_j(x) = \sum_{j=1}^J g_j(x) y_j \) and \( E_j(x) = \sum_{j=J+1}^\infty g_j(x) y_j \), so that \( F(x) = S_j(x) + E_j(x) \) for all \( J \). For all \( x \) and \( x' \) in \( U \),

\[
\| E_j(x) - E_j(x') \| \leq \| F(x) - S_j(x) \| + k_2 \| S_j(x) - S_j(x') \| \leq \| F(x) - F(x') \| + k \| F(x) - F(x') \| \leq k \| F(x) - F(x') \| ,
\]

for some constant \( k \).

Returning to the sequence \( \{ x_n \} \) in \( K \), we obtain the inequality

\[
k \| F(x_n) - F(x^*) \| \geq \| E_j(n)(x_n) - E_j(n)(x^*) \| \geq \| E_j(n)(x_n) \| - \| E_j(n)(x^*) \| \geq \epsilon_0 - \| E_j(n)(x^*) \| .
\]

The continuity of \( F \) yields a contradiction, hence (b) implies (c).

To show (c) implies (d), we must prove that \( \phi \circ F : U \to C \) is continuous for each \( \phi \in Y^* \). If \( x_n \to x \) in \( U \), then since \( \{ x_n \} \cup \{ x \} \) is compact, we have

\[
| \phi \circ F(x) - \phi \circ F(x_n) | = | \phi(S_j(x) + E_j(x)) - \phi(S_j(x_n) + E_j(x_n)) | \leq \| \phi \| \| S_j(x) - S_j(x_n) \| + \| \phi \| \| E_j(x) \| + \| E_j(x_n) \| .
\]

The right-hand side can be made arbitrarily small for sufficiently large \( n \), proving (c) implies (d).
To prove (d) implies (e), let \( x \in U \) and assume that for some sequence \( x_n \to x \), \( \|F(x_n)\| \to \infty \). Consider the evaluation mappings \( (F(x_n))^* \in Y^* \) and use the weak continuity to see that

\[
|F(x_n)|^*(\phi) \leq M = M(\phi)
\]

for all \( n \). The uniform boundedness principle implies that \( \|F(x_n)\|^* = \|F(x_n)\| \) cannot be arbitrarily large.

Finally, we show that (e) implies (a). Without loss of generality, we may assume that the norm of \( Y \) is given by \((\ast)\). Let \( \mathcal{S} \) be the linear subspace of \( Y^* \) spanned by the projection functionals \( \{e_j\} \). If \( \phi \) is any function in the closure of \( \mathcal{S} \), we claim that \( \phi \circ F : U \to \mathbb{C} \) is analytic. For this we must show that if \( x_0 \in U \), then \( \phi \circ F \) is analytic near \( x_0 \). Assume that \( \|F(x)\| \leq M \) for all \( x \in B_\rho(x_0) \subset U \). There is a sequence \( \sum \alpha_j e_j = \phi_0 \in \mathcal{S} \) such that \( \|\phi - \phi_0\| \to 0 \) as \( n \to \infty \); that is, given \( \varepsilon > 0 \), there is \( N \) such that

\[
\sup_{\|y\| \leq N} |\phi_n(y) - \phi(y)| < \varepsilon
\]

for all \( n \geq N \). In our situation, this yields \( |\phi_n(F(x)) - \phi(F(x))| < \varepsilon M \), if \( \|x - x_0\| < \delta \). Each \( \phi_n \circ F \) is analytic, and hence so too is \( \phi \circ F \).

We will prove next that \( \mathcal{S} \) is determining for \( Y \), which will imply (a). First, N. Dunford has shown that a function \( f \), mapping a domain in the complex plane into a Banach space \( Y \), is holomorphic if and only if \( \phi \circ f \) is holomorphic for all \( \phi \) in a determining manifold of \( Y^* \) [1, p. 354]. The required result then follows from the local boundedness of \( F \) [2, p. 112].

To prove that \( \mathcal{S} \) is determining, we let \( y = \sum_{j=1}^\infty \alpha_j y_j \in Y \). Letting \( \varepsilon > 0 \) be arbitrary, there is an integer \( J \) such that \( \|\alpha_j y_j\| > \|y\| - \varepsilon \). Consider the subspace of \( Y \) spanned by \( \{y_1, \ldots, y_J\} \), \( Y_J \), and the subspace of \( Y^* \) spanned by \( \{e_1, \ldots, e_J\} \), \( Y_J^* \). It is clear that by the Hahn Banach Theorem, we can consider \( T \in Y_J^* \) such that \( T(S_J) = \|S_J\| \), \( \|T\| = 1 \). Of course, we can consider \( T \) as a member of \( \mathcal{S} \) and this implies that \( \mathcal{S} \) is determining. This completes the proof.

To illustrate the sharpness of Theorem 1, we give the following example. Let \( F \) be the mapping from the unit disc \( \Delta \) of \( \mathbb{C} \) into \( l^2 \) defined as follows. Let \( z_n = \frac{1}{2} e^{i\theta_n} \), where \( \theta_n \to 0 \), \( \theta_n > 0 \). Choose a wedge \( B_n = \{re^{i\theta} : 0 < r < 1, |\theta - \theta_n| < \varepsilon_n\} \), where the \( \varepsilon_n \) are chosen so small that the \( B_n \) are disjoint. Let \( A_n \) be the complement of \( B_n \) in \( \Delta \). By the Runge approximation theorem, there are polynomials \( p_n \) which satisfy

\[
|p_n(z)| \leq \frac{1}{2} \quad \text{on} \quad A_n, \quad p_n(z_n) = 2.
\]

Define \( F(z) = (p_n(z))^* \). For all \( z \in \Delta \), \( F(z) \in l^2 \). However, \( \|F(z_n)\| \geq 2^n \), and so \( F \) is not locally bounded at \( \frac{1}{2} \). Hence, even though \( e_j \circ F(z) \) is a polynomial for all \( j \), \( F \) is not holomorphic.
3. Some comments and a problem. Theorem 1 was initially done for $l^p$ spaces, with cases for $p = \infty$, $1 < p < \infty$, and finally $p = 1$. Let $F: U \rightarrow l^p$ $(1 \leq p \leq \infty)$ with $F(x) = (g_j(x))$ and each $g_j: U \rightarrow \mathbb{C}$ holomorphic. If $x_0 \in U$ and $B_\delta(x_0) \subset U$, we can define the following set $C_n$ for each positive integer $n$:

$$C_n = \{x: \|x - x_0\| < \delta \text{ and } \|F(x)\| \leq n\}.$$ 

$C_n$ is closed and $\bigcup_{n=1}^{\infty} C_n = B_\delta(x_0)$. By the Baire category theorem, some $C_{n_0}$ has nonempty interior, and so, by Theorem 1, $F$ is analytic on $\text{int}(C_{n_0})$. Hence, there is an open dense subset of $U$ on which $F$ is analytic.

A result of Theorem 1(e) in the case where $X = \mathbb{C}$ and $Y = l^p$ $(1 \leq p < \infty)$ is that if $F: U \rightarrow l^p$ is holomorphic, then $F(z)$ is locally bounded. Therefore $\{g_j(z)\}$ is a normal family in the sense of Montel and $g_j(z) \rightarrow 0$ uniformly on compact subsets of $U$. We ask whether the normality of the sequence $\{g_j\}$ is sufficient to imply that $F: U \rightarrow l^p$ is holomorphic.

For $p = \infty$, the answer is affirmative. For, if $z_0 \in U$ and $|g_j(z)| \leq M$ on $B_\delta(z_0)$, then $|g_j'(z_0)| \leq M/\delta$. Hence, for all $z_0 \in U$, $\{g_j'(z_0)\} \in l^\infty$. The linear transformation $z \rightarrow (g_j'(z_0) \cdot z)$ is the Fréchet derivative of $F$ at $z_0$. To see this, consider

$$\left| \frac{g_j(z) - g_j(z_0)}{z - z_0} - g_j'(z_0) \right| = \left| \sum_{n=2}^{\infty} \frac{g_j^{(n)}(z_0)}{n!} (z - z_0)^{n-1} \right|$$

$$\leq |z - z_0| \sum_{n=2}^{\infty} \frac{M}{\delta^n} |z - z_0|^{n-2}.$$ 

The last sum is independent of $j$, which gives the result.

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Bibliography


Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506

Current address (Joseph A. Cima): Department of Mathematics, University of North Carolina, Chapel Hill, North Carolina 27514