PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, Number 1, November 1972

ON OSCILLATION OF COMPLEX LINEAR DIFFERENTIAL SYSTEMS

DONALD F. ST. MARY

ABSTRACT. This paper is concerned with first order linear matrix differential equations defined in the complex plane; such a system is said to be oscillatory in a domain D, if each component of a vector solution has a zero in D. It is shown that some sufficient conditions for nonoscillation on the real line, recently developed by Z. Nehari, can be extended to the plane.

1. The differential system

$$y' = Ay, \tag{1.1}$$

where $A=A(z)$ is an $n\times n$ matrix, and y an n-vector, the elements of which are holomorphic functions defined in a domain (open, simply connected set) D of the complex plane, is said to be *oscillatory* on D if there is a nontrivial solution $y=(y_1, \cdots, y_n)$ of (1.1) on D, each component of which takes the value zero at some point of D, i.e., $y_k(z_k)=0$, z_k in D, $k=1, \cdots, n$. The set of points z_k, $k=1, \cdots, n$, will be called a *zero set* of y. The purpose of this paper is to show that Z. Nehari's proof on the real line, of the theorem $\int_a^b \|A\| < \pi/2$ *implies nonoscillation on* $[a, b]$ [**1**], which is a generalization of the work of Kim [**2**], can be modified to yield the theorem in the complex plane. The resulting theorems encompass several known results.

2. We proceed to derive the basic inequalities, as in [**1**], which yield conditions for nonoscillation of the system (1.1).

THEOREM 2.1. *Let y and w be nontrivial solution vectors of the systems*

$$dy/dz = A(z)y, \tag{2.1}$$

$$dw/dz = B(z)w, \tag{2.2}$$

respectively, where the $n\times n$ matrices A, B are holomorphic in a domain D of the complex plane, and let Γ be a piecewise smooth simple arc in D

Presented to the Society, January 22, 1971; received by the editors August 26, 1971 and, in revised form, February 29, 1972.

AMS 1970 *subject classifications*. Primary 34A20, 34A30, 34C10.

Key words and phrases. Complex differential equations, first order matrix differential systems, oscillation, suborthogonal, disconjugacy, unitary matrix.

having end points a and b. If u, v are the unit vectors

$$u = y/\|y\|, \qquad v = w/\|w\| \tag{2.3}$$

and C is an arbitrary constant unitary matrix, then

$$\begin{aligned} |\arcsin\{\operatorname{Re}(u(b), Cv(b))\} - \arcsin\{\operatorname{Re}(u(a), Cv(a))\}| \\ \leqq \int_\Gamma (\|A\| + \|B\|)\,|dz|, \end{aligned} \tag{2.4}$$

where $\|A\|$ *denotes the norm,* $\sup_{\|\alpha\|=1} \|A\alpha\|$, (,) *is the usual inner product in* C_n, *and* Re x *denotes the real part of the complex number* x.

Furthermore, if Γ *contains a zero set of* y *or of* w *then*

$$\begin{aligned} |\arcsin\{\operatorname{Re}(u(b), Cv(b))\}| + |\arcsin\{\operatorname{Re}(u(a), Cv(a))\}| \\ \leqq \int_\Gamma (\|A\| + \|B\|)\,|dz|. \end{aligned} \tag{2.5}$$

Both (2.4) *and* (2.5) *hold with* Re *replaced by* Im, *the imaginary part.*

PROOF. Let the simple arc Γ have parametric representation, $z=z(t)$, $0 \leqq t \leqq 1$, with $z(0)=a$, $z(1)=b$. Without loss of generality we may assume Γ to be smooth. Differentiating (2.3) along this arc, we obtain

$$u' = y'/\|y\| - y[(y', y) + (y, y')]/2\,\|y\|^3,$$

and a similar expression for v', where "$'$" denotes d/dt, $0 \leqq t \leqq 1$. In view of (2.1) and (2.2), it follows that

$$u' = z'(t)Au - \tfrac{1}{2}u[(z'(t)Au, u) + (u, z'(t)Au)], \tag{2.6}$$

and

$$v' = z'(t)Bv - \tfrac{1}{2}v[(z'(t)Bv, v) + (v, z'(t)Bv)]. \tag{2.7}$$

Now

$$(u, Cv)' = (u', Cv) + (C^*u, v'), \tag{2.8}$$

$$(Cv, u)' = (v', C^*u) + (Cv, u') \tag{2.9}$$

and

$$2[\operatorname{Re}(u, Cv)]' = (u, Cv)' + (Cv, u)'. \tag{2.10}$$

Substituting (2.6)–(2.9) into (2.10) and simplifying we obtain

$$\begin{aligned} 2[\operatorname{Re}(u, Cv)]' &= \{(z'(t)Au, Cv - \bar{\alpha}u) + (Cv - \alpha u, z'(t)Au)\} \\ &\quad + \{(z'(t)Bv, C^*u - \bar{\beta}v) + (C^*u - \beta v, z'(t)Bv)\} \\ &= 2\operatorname{Re}(z'(t)Au, Cv - \{\operatorname{Re}(u, Cv)\}u) \\ &\quad + 2\operatorname{Re}(z'(t)Bv, C^*u - \{\operatorname{Re}(C^*u, v)\}v), \end{aligned} \tag{2.11}$$

where $\alpha=\frac{1}{2}[(u, Cv)+(Cv, u)]$, $\beta=\frac{1}{2}[(v, C^*u)+(C^*u, v)]$. Hence, since u and v are unit vectors,

$$|[\mathrm{Re}(u, Cv)]'| \leqq \|z'(t)A\| \cdot \|Cv - \{\mathrm{Re}(u, Cv)\}u\| + \|z'(t)B\| \cdot \|C^*u - \{\mathrm{Re}(C^*u, v)\}v\|$$

and thus

$$\|Cv - \{\mathrm{Re}(u, Cv)\}u\| = 1 - [\mathrm{Re}(u, Cv)]^2$$

and

$$\|C^*u - \{\mathrm{Re}(C^*u, v)\}v\| = 1 - [\mathrm{Re}(u, Cv)]^2$$

imply

$$|\{\mathrm{Re}(u, Cv)\}'|/(1 - \{\mathrm{Re}(u, Cv)\}^2) \leqq \|z'(t)A\| + \|z'(t)B\|.$$

Integration over the interval [0, 1] now yields (2.4). One obtains the result corresponding to (2.4) with Re replaced by Im in an analogous manner starting with $2i[\mathrm{Im}(u, Cv)]'=(u, Cv)'-(Cv, u)'$.

We now turn to (2.5). Suppose w has a zero set on Γ, then there is a set of points $t_1, \cdots, t_n$ in [0, 1] such that the ith component of $w(z(t))$ is zero at t_i, $i=1, \cdots, n$. The proof is now completely analogous to that of Nehari [**1**, p. 342]. ∎

3. Nehari's sufficient condition for nonoscillation on the real line can now be stated for the plane.

THEOREM 3.1. *Let Γ be a piecewise smooth simple arc in D. If for some holomorphic function $\mu=\mu(z)$ defined on D,*

$$\int_\Gamma \|A + \mu I\| \, |dz| < \pi/2, \tag{3.1}$$

then the system (2.1) *is nonoscillatory on Γ. The constant $\pi/2$ in* (3.1) *is the best possible.*

PROOF. First, the theorem is proved assuming $\mu\equiv 0$. If (2.1) is oscillatory on Γ then Γ contains a zero set of a nontrivial solution y. Now, one applies (2.5) with $B=0$, $C=I$ and $v(z)$ taken to be the constant unit vector $u(b)$, to obtain

$$\pi/2 + \arcsin |\mathrm{Re}(u(a), u(b))| \leqq \int_\Gamma \|A\| \, |dz|.$$

The general case is taken care of as in Nehari's Theorem 2.2 [**1**]; also the example provided there demonstrates that $\pi/2$ is the best possible constant. ∎

Kim [**2**, Theorem 2.1] obtained the value 1 instead of $\pi/2$ in a similar theorem.

The complex formulation, Theorem 2.1, of Nehari's Theorem immediately yields a new degree of freedom in its application as demonstrated by the following theorem.

THEOREM 3.2. *Let Γ be a piecewise smooth, simple, closed curve in D. If*

$$\int_{\Gamma} \|A\| \, |dz| < \pi$$

then the system (2.1) *is nonoscillatory on* Γ. *The constant* π *is best possible.*

PROOF. If the curve Γ contains a zero set of a solution y then one uses (2.5) with $a=b$ any point on Γ and, as before, $v\equiv u(a)$ to obtain $\int_{\Gamma} \|A\| \geqq \pi$. Trivial modifications of Nehari's example [**1**, Theorem 2.2] yield π best possible. ∎

We remark that using Theorem 2.1 it is possible to obtain theorems like those above involving the concept of suborthogonality.

Finally, the refinement of the work of Kim [**2**] by Schwarz [**3**], in particular [**3**, Lemma 3] and the use of invariance under conformal mappings, makes it possible to obtain results analogous to those announced by Nehari [**4**].

REFERENCES

1. Z. Nehari, *Oscillation theorems for systems of linear differential equations*, Trans. Amer. Math. Soc. **139** (1969), 339–347. MR **39** #542.

2. W. J. Kim, *Disconjugacy and disfocality of differential systems*, J. Math. Anal. Appl. **26** (1969), 9–19. MR **38** #4759.

3. B. Schwarz, *Norm conditions for disconjugacy of complex differential systems*, J. Math. Anal. Appl. **28** (1969), 553–568. MR **40** #2963.

4. Z. Nehari, *Oscillation criteria for systems of linear differential equations*, Notices Amer. Math. Soc. **18** (1971), 370. Abstract #683-B26.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01002