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ANALYTIC SOLUTIONS OF A NEUTRAL DIFFERENTIAL EQUATION NEAR A SINGULAR POINT

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ABSTRACT. Fixed point techniques are employed to prove existence and uniqueness of a holomorphic solution to a functional differential equation of neutral type in the neighborhood of a regular singular point.

1. A great deal of progress has been made in the general theory of functional differential equations in the past decade. However, the analytic theory has received little attention, because in the case where the initial set is nontrivial, i.e., does not reduce to a single point, analyticity and growth conditions alone will not in general guarantee the existence of an analytic solution to the fundamental initial value problem, see [5], [6]. Indeed, an arbitrary initial value problem will have an analytic solution if and only if the analytic extension of the initial function satisfies the differential equation.

The problem of analytic extension does not arise if the initial set reduces to a single point. If the initial point is an ordinary point for the differential equation, the fundamental analytic initial value problem will have analytic solutions under appropriate regularity conditions. For example, the initial value problem for the neutral differential equation

$$x'(t) = f(t, x(t), x(g(t)), x'(h(t))); \qquad x(0) = x_0,$$

where f, g, and h are holomorphic functions, and $g(0)=h(0)=0$; $|g'(t)|$, $|h'(t)|\leqq 1$, has a unique holomorphic solution in a region about the origin, provided that f satisfies a suitable growth condition, see [2].

In the case when the origin is a regular singular point (and the initial set consists of the origin alone), El'sgol'c [1] has studied a linear functional differential equation analogous to the Euler equation; Martynjuk [4] has used power series methods to treat a larger class of linear equations. In a recent paper, Grudo [3] has also used power series to study nonlinear equations where the deviating arguments have the form $g(t)=pt$, where p

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is a constant of modulus between 0 and 1. In this note we use fixed point techniques to prove existence and uniqueness of a holomorphic solution for an equation of neutral type in the neighborhood of a regular singular point.

2. Let t be a complex variable, let α, β, γ, K and M be positive constants with $\alpha<\beta/M$, $K\leqq\gamma/\beta$. Let $I=\{t:|t|<\alpha\}$, $\Omega=\{(t, x, y, z):t\in I, |x|<\beta, |y|<\gamma, |z|<M|t|\}$, $U=\{(t, x):t\in I, |x|<\beta\}$. Consider the initial value problem for the neutral differential equation

$$tx'(t)=f(t, x(t), x(g(t, x(t))), h(t)x'(h(t))), \tag{1}$$

$$x(0)=0. \tag{2}$$

Here it is assumed that

(i) $f(t, x, y, z)$ is holomorphic in Ω and continuous on $\bar{\Omega}$, with $f(0, 0, 0, 0)=0$.

(ii) g is holomorphic in U and continuous on $\bar{U}$ with $g(U)\subseteq I$, h is holomorphic in I and continuous on $\bar{I}$, with $g(0, 0)=h(0)=0$, and $\sup_U|g_t|+M\sup_U|g_x|\leqq K$, $\sup_I|h'|\leqq 1$.

(iii) There exist nonnegative constants L_1, L_2, L_3 with $L_1+KL_2+L_3<1$, such that $\sup_\Omega|f_x|\leqq L_1$, $\sup_\Omega|f_y|\leqq L_2$, $\sup_\Omega|f_z|\leqq L_3$, and $\sup_\Omega|f_t|\leqq M(1-L)$.

THEOREM 1. *Let the hypotheses* (i)–(iii) *be satisfied. Then the problem* (1)–(2) *has a solution $x(t)$ holomorphic in I, satisfying $|x'(t)|\leqq M$.*

PROOF. Let X be the B-space of functions holomorphic in I and continuous on $\bar{I}$, with uniform norm. Let $S=\{z\in X:|z(t)|\leqq M|t|$, for all $t\in\bar{I}\}$. For $z\in S$, let $Tz(t)=f(t, I(z, t), I(z, g(t, I(z, t))), z(h(t)))$, where $I(z, t)=\int_0^t(z(s)/s)\,ds$. Note that $I(z, t)$ is continuous on S. Indeed, let $\varepsilon>0$, $t=\rho e^{i\theta}$, for $\rho>0$, and set $e^{i\theta}=w$. For $z_1, z_2\in S$,

$$\begin{aligned}|I(z_1, t)-I(z_2, t)| &= \left|\int_0^t\frac{z_1(s)-z_2(s)}{s}\,ds\right| \\ &\leqq \left|\int_0^{\varepsilon w/4M}\frac{z_1(s)-z_2(s)}{s}\,ds\right|+\left|\int_{\varepsilon w/4M}^t\frac{z_1(s)-z_2(s)}{s}\,ds\right| \\ &\leqq \frac{\varepsilon}{2}+\frac{4M\alpha}{\varepsilon}|z_1-z_2|.\end{aligned}$$

Thus, if $|z_1-z_2|<\varepsilon^2/8M\alpha$, then $|I(z_1, t)-I(z_2, t)|<\varepsilon$.

This, together with the hypotheses on f, g and h, ensures that T is a continuous map of S into S. By continuity of f, for each $\varepsilon>0$, there exists

$\delta(\varepsilon)>0$ such that for all $z\in S$, if $t_1, t_2, t\in \bar{I}$ with $|t_1-t_2|\leqq\delta(\varepsilon)$, then

$$|f(t_1, I(z, t_1), I(z, g(t_1, I(z, t_1))), z(h(t))) \\ - f(t_2, I(z, t_2), I(z, g(t_2, I(z, t_2))), z(h(t)))| < \varepsilon.$$

Let $S_\varepsilon=\{z\in S:|z(t_1)-z(t_2)|\leqq\varepsilon/(1-L)$ for all $t_1, t_2\in \bar{I}, |t_1-t_2|\leqq\delta(\varepsilon)\}$. If $z\in S_\varepsilon$, and if $t_1, t_2\in\bar{I}$ with $|t_1-t_2|\leqq\delta(\varepsilon)$, then $|Tz(t_1)-Tz(t_2)|\leqq\varepsilon+\varepsilon L/(1-L)=\varepsilon/(1-L)$. Hence $TS_\varepsilon\subseteq S_\varepsilon$. Let $S_0=\bigcap_{\varepsilon>0}S_\varepsilon$. Then S_0 is nonempty, compact and convex, and $TS_0\subseteq S_0$. Thus by Schauder's theorem, T has a fixed point $z(t)$ in S_0. Setting $z=tx'$ and integrating the holomorphic function $z(t)/t$ from zero to t completes the proof.

REMARK. Hypothesis (iii) may be replaced by

(iiia) $\sup_{\Omega_{t_0}}|f|\leqq M|t_0|$, for each $t_0\in\bar{I}$, where Ω_{t_0} is the section of Ω at t_0, and there exists a constant L, $0\leqq L<1$, such that $|f_z|\leqq L$ in Ω.

THEOREM 2. *If, in addition to the hypotheses* (i)–(iii), *the following is satisfied*:

(iv) $L+L_2M\sup_U|g_x|\leqq 1$,

the solution whose existence was established in Theorem 1 *is unique in the class of functions* $x(t)$ *holomorphic and satisfying* $|x'(t)|\leqq M|t|$ *in* I.

PROOF. Let X_0 be the B-space of functions $z(t)$ holomorphic in I, continuous on $\bar{I}$, such that $z(0)=0$, with the norm $|||z(t)|||=\sup_{s\in I}|z(s)/s|$. Let $S_*=\{z\in X_0:|z(t)|\leqq M|t|\}$. Define $T:S_*\to S_*$ as in the proof of Theorem 1. The hypotheses of Theorem 2 guarantee that T is a contraction on S_*. Thus, by the Banach fixed point theorem, T has a fixed point in S_*. The remainder of the proof is carried out as in Theorem 1.

REMARK. The following example shows that Theorem 2 is sharp. Consider the problem

$$tx'(t) = c_1x(t) + c_2x(t) + c_3tx'(t), \tag{3}$$

$$x(0) = 0, \tag{2}$$

where c_1, c_2 and c_3 are nonnegative real constants with $c_3<1$. Equation (3) is of the form (1) with $g(t, x)=t$, $h(t)=t$. Then $K=1$, $L=c_1+c_2+c_3$. Equation (3) can be written in the form $(1-c_3)tx'(t)=(c_1+c_2)x(t)$ or $tx'(t)=((c_1+c_2)/(1-c_3))x(t)$.

If c is a real constant, the problem

$$tx'(t) = cx(t), \qquad x(0) = 0,$$

has the unique holomorphic solution $x(t)\equiv 0$ if $0\leqq c<1$, while if $c=1$, the problem has the holomorphic solution $x(t)=t$ in addition to the trivial solution. Hence if $L<1$, the holomorphic solution of (3)–(2) is unique, while uniqueness fails if $L=1$. It is clear that in this problem M may be chosen arbitrarily large.

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