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AN EXAMPLE OF A WILD $(n-1)$-SPHERE IN S^n IN WHICH EACH 2-COMPLEX IS TAME[1]

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ABSTRACT. The main purpose of this note is to give an example promised in the title (for $n \geqq 5$). The example is the k-fold suspension ($k \geqq 2$) of Bing's 2-sphere in S^3 in which each closed, nowhere dense subset is tame. Our efforts were motivated by recent results of Seebeck and Sher concerning tame cells in wild cells and spheres. In fact, we are able to strengthen one of Seebeck's results in order to prove that every embedding of an m-dimensional polyhedron in our wild $(n-1)$-sphere S $(n-m \geqq 3)$ can be approximated in S by an embedding that is tame in S^n.

The purpose of this note is to give an example supplementing some results in recent papers of Seebeck [**9**] and Sher [**10**]. Specifically, let $\Sigma \subset E^3$ be the 2-sphere constructed by Bing in [**1**]. We shall prove that Σ has the following property.

THEOREM 1. *If K is a 2-complex topologically embedded in $\Sigma \times E^k \subset E^3 \times E^k$ $(k \geqq 2)$, then K is tame in $E^3 \times E^k$.*

COROLLARY 1.1. *There is an $(n-1)$-sphere S in S^n $(n \geqq 5)$ having the property that S is everywhere wild and every 2-complex topologically embedded in S is tame in S^n.*

PROOF. Take S to be the $(n-3)$-fold suspension of Σ, $\Sigma^{n-3}(\Sigma)$, in $S^n = \Sigma^{n-3}(S^3)$ $(n \geqq 5)$. Then every 2-complex in S will be locally tame modulo its intersection with the suspension $(n-4)$-sphere. But such a 2-complex must have a 1-ULC complement in S^n and, hence, is tame by [**5**].

The particular property of the 2-sphere Σ in E^3 that we wish to use is the following:

If X is a closed, nowhere dense subset of Σ and if $\varepsilon > 0$, then there exist a tame 2-sphere S in E^3 and an ε-homeomorphism $f: \Sigma \to S$ such that $f(x) = x$ if $x \in X$. (See the proof of Theorem 1 of [**7**].)

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For definitions and background material, the reader is referred to [9].

DEFINITION. Suppose that $A \subset B$ are subsets of a metric space Y. We say that *A is* 1-LC *in B at a point $p \in Y$* iff for each $\varepsilon > 0$ there corresponds $\delta > 0$ such that if Γ is a loop in $N_\delta(p) \cap A$, then Γ is homotopic to a point in $N_\varepsilon(p) \cap B$. ($N_\varepsilon(p)$, of course, denotes the ε-neighborhood of p in Y.)

PROPOSITION 1. *Suppose that X is a k-dimensional compactum in an m-manifold M and M is topologically embedded as a closed subset of an n-manifold N ($n-k \geqq 3$) in such a way that $N-M$ is* 1-LC *in $N-X$ at each point of X. Then $N-X$ is* 1-LC *at each point of X.*

PROOF. If $m < n-1$, then the proposition is trivial, since any loop in N may be homotopied off of M by an arbitrarily small homotopy. We shall, therefore, assume that $m = n-1$.

Observe that since X is compact, it must satisfy the hypothesis of the proposition uniformly; that is, for each $\varepsilon > 0$ there exists $\delta > 0$ such that any loop in $N-M$ of diameter less than δ bounds a singular disk in $N-X$ of diameter less than ε.

Suppose now that Γ' is a small loop in $N-X$. Then Γ' is homotopic in $N-X$ to a simple closed curve Γ in $N-X$ (assume $n \geqq 3$) such that $\dim(\Gamma \cap M) \leqq 0$. Let $C = \Gamma \cap M$. Since $k \leqq n-3$, there exists an arc A in $\dot{M} - X$ containing C.

Let α be a component of $\Gamma - M$ with endpoints a and b in M (i.e., $\{a, b\} = \bar{\alpha} \cap M$), and let β be the subarc of A joining a and b. Let Δ be the standard 2-simplex with edges e_1, e_2, and e_3 and let $f: \mathrm{Bd}\,\Delta \to \alpha \cup \beta$ be a map such that $f(e_1) = \beta$ and $f(e_2 \cup e_3) = \bar{\alpha}$. Let T be a locally finite triangulation of $\Delta - e_1$ with 2-simplexes $\Delta_1, \Delta_2, \Delta_3, \cdots$ having the property that $\operatorname{diam} \Delta_i \to 0$ as $i \to \infty$. Let $K = |T^1| \cup e_1$. (T^1 = the 1-skeleton of T.) Since $N-M$ is 0-LC at points of M (hence, uniformly on a compact subset of M), the map $f: \mathrm{Bd}\,\Delta \to N$ can be extended to a map $f_1: K \to N$ such that $f_1(|T^1|) \cap M = \varnothing$.

By the hypothesis, for each $i = 1, 2, \cdots, f_1 | \mathrm{Bd}\,\Delta_i : \mathrm{Bd}\,\Delta_i \to N-M$ can be extended to $f_2^i: \Delta_i \to N-X$ in such a way that $\operatorname{diam} f_2^i(\Delta_i) \to 0$ as $i \to \infty$. Thus $f: \mathrm{Bd}\,\Delta \to N-X$ extends to a map $\bar{f}: \Delta \to N-X$.

Write $\Gamma - M = \alpha_1 \cup \alpha_2 \cup \cdots$, where each α_i is a component of $\Gamma - M$, and let β_i be subarc of A joining the endpoints of α_i. Then $\operatorname{diam}(\alpha_i \cup \beta_i) \to 0$ as $i \to \infty$. Hence, by the above construction, $\alpha_i \cup \beta_i$ bounds a singular disk D_i in $N-X$, which can be chosen so that $\operatorname{diam}(D_i) \to 0$ as $i \to \infty$. Thus Γ is homotopic in $N-X$ to a map of Γ into A, and hence Γ is homotopic to a point in $(\bigcup_{i=1}^\infty D_i) \cup A$. Clearly the diameter of $\bigcup_{i=1}^\infty D_i \cup A$ can be made small, depending on the size of Γ. (A precise description of a construction quite similar to this may be found in the proof of Lemma 8 of [8].)

PROPOSITION 2. *Let Σ be the 2-sphere in E^3 described previously and let X be a closed, nowhere dense subset of Σ. Then $E^3-\Sigma$ is* 1-LC *in E^3-X at each point of X.*[2]

PROOF. By [**2**, Proposition 1], there exists for each $\delta>0$ an open set V containing Σ and $\eta>0$ such that if $h:\Sigma\to V$ is an embedding that moves no point farther than η, then V retracts onto $h(\Sigma)$ via a retraction $r: V\to h(\Sigma)$ that moves no point of V farther than δ.

Given $\varepsilon>0$, let $V\supset\Sigma$ and $\eta>0$ correspond to $\varepsilon/2$ as above. Given $p\in X$, choose $\delta>0$ such that $\delta\leqq\varepsilon/2$ and $N_\delta(p)\subset V$. Now suppose that Γ is a loop in $N_\delta(p)-\Sigma$. Then there exists an embedding $h:\Sigma\to V$ such that $d(x, h(x))<\eta/2$ for all $x\in\Sigma$, $h|X=$inclusion, $h(\Sigma)$ is tame, and $h(\Sigma)\cap\Gamma=\varnothing$. Since $h(\Sigma)$ is tame, there exists an $(\eta/2)$-homeomorphism h' of E^3 such that $h'h(\Sigma)\subset V-(X\cup\Gamma)$ and $h'h(\Sigma)$ separates X and Γ. Let D be a singular disk in $N_\delta(p)$ with boundary Γ, and write $V-h'h(\Sigma)=V_1\cup V_2$, where V_1 and V_2 are disjoint open sets containing Γ and X, respectively. Let $r': V\to V$ be defined by $r'(y)=y$ if $y\in V_1\cup h'h(\Sigma)$ and $r'(y)=r(y)$ if $y\in V_2$. Then $r'(D)$ is a singular disk in $N_\varepsilon(p)-X$ with boundary Γ.

PROPOSITION 3. *Suppose that X is a* 1*-dimensional compactum in $\Sigma\times E^k\subset E^3\times E^k$. Then $(E^3\times E^k)-X$ is* 1-ULC.

PROOF. By Proposition 1, it is sufficient to prove that $(E^3\times E^k)-(\Sigma\times E^k)$ is 1-LC in $(E^3\times E^k)-X$ at each point of X. Suppose $p\in X$. Then $p=(z, w)$ where $z\in\Sigma$ and $w\in E^k$. Let $X'=X\cap(E^3\times\{w\})$ and let π be the projection of $E^3\times E^k$ onto $E^3\times\{w\}$.

Suppose that Γ is a loop in a neighborhood of p such that $\Gamma\cap(\Sigma\times E^k)=\varnothing$. Then Γ is homotopic in $(E^3\times E^k)-(\Sigma\times E^k)$ to the projection $\pi(\Gamma)$ of Γ into $E^3\times\{w\}$. By Proposition 2, $\pi(\Gamma)$ can be made to bound a small singular disk in $(E^3\times\{w\})-X'$. Q.E.D.

PROPOSITION 4. *Suppose that $A\subset Y$ has the absolute homotopy extension property* (*see* [**6**]) *and that A is connected. Then every homeomorphism $h:A\times E^1\to A\times E^1$ that commutes with the projection $\pi_1:A\times E^1\to A$ can be extended to a homeomorphism $H: Y\times E^1- Y\times E^1$ that commutes with the projection $\pi_1: Y\times E^1\to Y$.*

PROOF. Let $\mathscr{H}(E^1)$ (resp. $\mathscr{H}^+(E^1)$) be the space of all homeomorphisms (resp. all orientation preserving homeomorphisms) of E^1 onto E^1 (with the compact open topology). Let $\pi_2: Y\times E^1\to E^1$ be the projection. Given $h:A\times E^1\to A\times E^1$ as above, define $\phi:A\to\mathscr{H}(E^1)$ by

$$[\phi(a)](t) = \pi_2 h(a, t).$$

[2] Cannon announced a general theorem in [Notices Amer. Math. Soc. **17** (1970), 469, Abstract #70T-G43] that implies Proposition 2.

Since A is connected, we may assume that $\phi(A) \subset \mathscr{H}^+(E^1)$. Since $\mathscr{H}^+(E^1)$ is contractible, ϕ is homotopic, via ϕ_s ($s \in [0, 1]$), to $i: A \to \mathscr{H}^+(E^1)$, where $i(a) = \text{identity}: E^1 \to E^1$ for each $a \in A$. Extend to $\bar{i}: Y \to \mathscr{H}^+(E^1)$, where $\bar{i}(y) = \text{identity}$ for each $y \in Y$. Since A has the AHEP in Y, there exists a homotopy $\bar{\phi}_s: Y \to \mathscr{H}^+(E^1)$ ($s \in [0, 1]$) such that $\bar{\phi}_1 = \bar{i}$ and $\bar{\phi}_s | A = \phi_s$ for each $s \in [0, 1]$. Then $H: Y \times E^1 \to Y \times E^1$ defined by

$$H(y, t) = (y, (\bar{\phi}_0(y))(t))$$

is the desired homeomorphism of $Y \times E^1$.

PROOF OF THEOREM 1. Suppose K is a 2-complex topologically embedded in $\Sigma \times E^k \subset E^3 \times E^k$ ($k \geqq 2$). Write E^k as $E^{k-1} \times E^1$. There exists a homeomorphism $h: \Sigma \times E^{k-1} \times E^1 \to \Sigma \times E^{k-1} \times E^1$ that commutes with the projection of $\Sigma \times E^{k-1} \times E^1$ onto $\Sigma \times E^{k-1}$ such that $\dim[h(K) \cap (\Sigma \times E^{k-1} \times \{w\})] \leqq 1$ for each $w \in E^1$. (To get h, take a countable dense subset $\{x_1, x_2, x_3, \cdots\}$ of K and get a sequence of homeomorphisms $h_1, h_2, \cdots$ of $\Sigma \times E^{k-1} \times E^1$ onto itself, each commuting with the projection onto $\Sigma \times E^{k-1}$, converging to a homeomorphism h of $\Sigma \times E^{k-1} \times E^1$ onto itself such that the projection of the set $\{h(x_1), h(x_2), \cdots\}$ into E^1 is one-to-one. Since a 2-dimensional subset of K must contain an open set, and hence more than one x_i, $\dim(h(K) \cap \Sigma \times E^{k-1} \times \{w\})$ must be $\leqq 1$ for each $w \in E^1$.)

Observe that $E^3 \cup (\Sigma \times I)$ with Σ identified with $\Sigma \times 0$ is a retract of $E^3 \times I$. Hence, $(E^3 \times E^{k-1}) \cup (\Sigma \times E^{k-1} \times I)$ is a retract of $E^3 \times E^{k-1} \times I$, and so $\Sigma \times E^{k-1}$ has the absolute homotopy extension property in $E^3 \times E^{k-1}$. Thus we may apply Proposition 4 to extend $h: \Sigma \times E^{k-1} \times E^1 \to \Sigma \times E^{k-1} \to E^1$ to $H: E^3 \times E^{k-1} \times E^1 \to E^3 \times E^{k-1} \times E^1$. We will then have both

$$\dim[H(K) \cap (E^3 \times E^{k-1} \times \{w\})] \leqq 1$$

for each $w \in E^1$ and

$$H(K) \subset (\Sigma \times E^{k-1} \times E^1).$$

Now by Proposition 3, $(E^3 \times E^{k-1} \times \{w\}) - K$ is 1-ULC for each $w \in E^1$, so we may apply Theorem 1 of [3] to conclude that $(E^3 \times E^k) - K$ is 1-ULC. Therefore, K is tame in $E^3 \times E^k$ by [5].

We conclude this note with two additional theorems. The first is more or less a corollary to the proof of Theorem 1. The second is an extension of Theorem 4 of [9]. We follow it with an application to our example.

THEOREM 2. *Let S be the $(n-3)$-fold suspension of Σ in $S^n = \Sigma^{n-3}(S^3)$ ($n \geqq 5$). Then every m-dimensional polyhedron P ($m \leqq n-3$) that is piecewise linearly embedded in S is tame in S^n.*

PROOF. By the reasoning in the proof of Corollary 1.1, it suffices to consider $P \subset \Sigma \times E^k \subset E^3 \times E^k$ ($k = n-3$). By [5], we may assume that P is an m-simplex linearly embedded in $\Sigma \times E^k$. Let $\{v_0, \cdots, v_m\}$ be the vertices of P. Apply Proposition 4 m times to get a homeomorphism H of $E^3 \times E^k$ such that H commutes with the projection onto E^3, $H|\Sigma \times E^k$ is linear on P, and the points $\pi_2 H(v_0), \cdots, \pi_2 H(v_m)$ are independent in E^k. Then $\dim[H(P) \cap (\Sigma \times \{w\})] \leqq 0$ for each $w \in E^k$. By the method of proof of Theorem 1, we see that $(E^3 \times E^k) - H(P)$ is 1-ULC, and hence P is tame in $E^3 \times E^k$.

THEOREM 3. *Suppose that N is a* PL *n-manifold ($n \geqq 6$) and M is a* PL *$(n-1)$-manifold topologically embedded in N such that every 2-complex of M can be approximated by a 2-complex in M that is tame in N. Then each k-dimensional polyhedron P topologically embedded in M ($n-k \geqq 3$) can be approximated in M by embeddings that are tame in N.*

COROLLARY 3.1. *Suppose that $S \subset S^n$ ($n \geqq 5$) is as in Theorem 2. Then every k-dimensional polyhedron P ($n-k \geqq 3$) topologically embedded in S can be approximated in S by embeddings that are tame in S^n.*

PROOF OF THEOREM 3. We follow closely the proof of Theorem 4 of **[9]**. Let $\{K_i\}$ and $\{L_i\}$ be sequences of barycentric subdivisions of triangulations of N and M, respectively. As in the proof of Theorem 4 of **[9]**, we need to show that if $P \subset M$ is a k-dimensional polyhedron ($n-k \geqq 3$) then there exists, for each $\varepsilon > 0$, an ε-homeomorphism $h: M \to M$ such that $h(P) \cap (\bigcup_{i=1}^{\infty} |K_i^2|) = \varnothing$ ($K_i^2 =$ 2-skeleton of K_i). This will imply that $N-P$ is 1-ULC and, hence, that P is tame in N by **[5]**.

As in **[9]**, we can construct an $(\varepsilon/2)$-homeomorphism $f: N \to N$ such that $f(\bigcup |K_i^0|) \cap M = \varnothing$ and $f(\bigcup |K_i^2|) \cap (\bigcup Q_i) = \varnothing$, where Q_i is an approximation of $|L_i^2|$ that is tame in N. Since $(\bigcup |K_i^0|) \cap |K_j^2|$ is dense in $|K_j^2|$ for each j, it must be that $\dim[f(|K_j^2|) \cap M] \leqq 1$ for each j. Let $X_j = f(|K_j^2|) \cap M$. Then X_j is a 1-dimensional compactum in M and $M - X_j$ is 1-ULC for each j. (This is true because $X_j \cap (\bigcup Q_i) = \varnothing$ for each j.)

Suppose now that P is a k-dimensional polyhedron in M with $k \leqq n-3$. By the Corollary in **[4]** there exists, for every sequence of positive numbers $\varepsilon_1, \varepsilon_2, \cdots$, a sequence $P_1, P_2, \cdots$ of 1-dimensional subpolyhedra of M, each of which we may assume to be disjoint from P (since $\dim P \leqq \dim M - 2$), such that X_j can be engulfed by the open set $M-P$ containing P_j via an ε_j-homeomorphism h_j of M.

Thus we may inductively construct a sequence $\{\varepsilon_j\}$ of positive numbers and a sequence $\{h_j\}$ of homeomorphisms of M onto M such that

(i) $\lim_{j \to \infty} h_j \cdots h_2 h_1 = h_0$ is an $(\varepsilon/2)$-homeomorphism of M onto M, and

(ii) $h_0(X_j) \cap P = \varnothing$ for each j.

Thus $h = f^{-1} h_0^{-1}: M \to M$ is an ε-homeomorphism and $h(P) \cap (\bigcup |K_i^2|) = \varnothing$.

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