AN APPROXIMATION THEOREM FOR INFINITE GAMES

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Abstract. We consider infinite, two person zero sum games played as follows: On the nth move, players A, B select privately from fixed finite sets, \( A_n, B_n \), the result of their selections being made known before the next selection is made. A point in the associated sequence space \( \Omega = \prod_{n=1}^{\infty} (A_n \times B_n) \) is thus produced upon which B pays A an amount determined by a payoff function defined on \( \Omega \). We show that if the payoff functions of games \( \{G_n\} \) are upper semicontinuous and decrease pointwise to a function which is the payoff for a game, \( G \), then \( \text{Val}(G_n) \downarrow \text{Val}(G) \). This shows that a certain class of games can be approximated by finite games. We then give a counterexample to possibly more general approximation theorems by displaying a sequence of games \( \{G_n\} \) with upper semicontinuous payoff functions which increase to the payoff of a game \( G \), and where \( \text{Val}(G_n) = 0 \) for all \( n \) but \( \text{Val}(G) = 1 \).

Introduction. Infinite games with imperfect information have been studied by several writers, notably Blackwell [1], [2], Gillette [3], Milnor and Shapley [4].

Before proceeding with the main result we will introduce notation and describe the structure of these games.

Let \( \{A_n\}, \{B_n\} \) be sequences of nonempty finite sets. Let \( Z_n = A_n \times B_n \) and let \( \Omega \) be the space \( \prod_{n=1}^{\infty} Z_n \) of infinite sequences \( \omega = (z_1, z_2, \cdots) \) where \( z_n \in Z_n \). Let \( X = \{ (z_1, z_2, \cdots, z_n) | z_i \in Z_i, n=1, 2, \cdots \} \) be the set of finite starting sequences or partial histories.

Suppose \( f \) is a bounded Baire function on \( \Omega \) (with respect to the product topology). Then \( f \), called a payoff function, defines a zero-sum two person game \( G_f \), played as follows:

First, player A selects \( a_1 \in A_1 \) while player B simultaneously selects...
b₁ ∈ B₁. The result, z₁ = (a₁, b₁) ∈ Z₁, is announced to both players, upon which A selects a₂ ∈ A₂ while B is selecting b₂ ∈ B₂, etc. The result of this infinite sequence of moves is a point ω = (z₁, z₂, ⋅ ⋅ ⋅) ∈ Ω and B pays A the amount f(ω). For any partial history x ∈ X we can define a subgame of the original game (usually referred to as the original game, "starting from x") by having the players play as above except redefining the payoff function as fₙ(ω) = f(xₙω).

A strategy α (β) for A (B) gives for each partial history x (of length n, say) a probability distribution on Aₙ₊₁ (Bₙ₊₁) with the stipulation that if the current position is x, A (B) will make his next choice according to α (β). A pair of strategies, (α, β) defines a probability distribution, Pₙ on Ω and, hence, an expected payoff to A in G when A uses α and B uses β:

\[ E(f, \alpha, \beta) = \int f(\omega) \, dPₙ(\omega). \]

(We will usually omit the α, β from the notation when it is clear what is happening.)

The lower and upper values of G are, respectively,

\[ L(G) = \inf_{x} \sup_{\beta} E(f, \alpha, \beta), \quad U(G) = \sup_{x} \inf_{\beta} E(f, \alpha, \beta). \]

It is always true that L(G) ≤ U(G); if L(G) = U(G), this common value is called the value of G, and will be denoted by Val(G).

Finally, a payoff function f is called upper (lower) semicontinuous if \( \liminf_{n \to \infty} f(\omega_n) \leq f(\omega) \) (\( \limsup_{n \to \infty} f(\omega_n) \geq f(\omega) \)).

The result of [5] we will use is as follows. Let M be compact, N any space, f defined on M × N which is concave-convexlike. If f(μ, ν) is u.s.c. in μ for each ν, then \( \sup_\mu \inf_\nu f = \inf_\nu \sup_\mu f \). We show how to apply this to the present situation: The space of plays, Ω, and the set of strategies for each player gives rise to a product of compact spaces, \( \Omega_{A}^* \times \Omega_{B}^* \), where \( \Omega_{A}^* = \prod_{n=1}^{\infty} A_n^*, \Omega_{B}^* = \prod_{n=1}^{\infty} B_n^* \). We define \( A_n^*, B_n^* \) as follows: If α is a strategy for player A, the corresponding member of \( \Omega_{A}^* \) is a sequence \( (\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots) \), where \( \alpha_n \in A_n^* \) is a finite list of probabilities on \( A_n \), one for each possible past history (the list is finite since the sets \( A_n, B_n, n=1, 2, \cdots \), are finite). \( B_n^* \) is defined analogously. The corresponding product topology makes \( \Omega_{A}^*, \Omega_{B}^* \) compact. If f is a payoff on Ω, we get a corresponding payoff \( f^* \) on \( \Omega_{A}^* \times \Omega_{B}^* \) by defining \( f^*(\alpha, \beta) = E(f, \alpha, \beta) \). If f is u.s.c. on Ω, so is \( f^* \) on \( \Omega_{A}^* \times \Omega_{B}^* \) (in the product topology); also \( f^* \) is linear, so [5] applies. It is easily seen that \( \sup_\alpha \inf_\beta f^* = \inf_\beta \sup_\alpha f^* \) implies the game with payoff f has a value, so [5] gives us that games with u.s.c. payoffs have a value.
We shall now prove the main result, namely

**Theorem 2.1.** Suppose $G_{f_n}$ are games with upper semicontinuous payoff functions $f_n$, where the $f_n$ are (pointwise) nonincreasing, $f_n \downarrow f$ (which is, therefore, also upper s.c.). Then $\text{Val}(G_f) = \lim_n \text{Val}(G_{f_n})$.

We first prove a lemma.

**Lemma 2.1.** Suppose $f_n, f$ are as above. For any partial history $x$, let $m_x = \lim_n \text{Val}_x(G_{f_n})$, where $\text{Val}_x(G_{f_n})$ means the value of the game with payoff $f_n$, starting from $x$. (The value of games with upper s.c. payoff function exists, by [5].) Let $G^*_x$ be the one move game which starts at $x$ and has payoff $g = m_x$ if $y$ is the next position hit. Claim $\text{Val}(G^*_x) \geq m_x$.

**Proof of Lemma.** We will show by contradiction that for fixed $\varepsilon > 0$, $A$ can play in $G^*_x$ to guarantee that $E(g) \leq m_x - \varepsilon$. Assume not; then for every strategy of $A$, player $B$ can play to make $E(g) < m_x - \varepsilon$.

For each possible next position, $y_1, y_2, \ldots, y_k$, let $f_{n_i}$ be such that $\text{Val}_{y_i}(G_{f_{n_i}}) < m_{y_i} + \varepsilon/2$. Let $m = \max_i(n_i)$; so that for all $i$,

$$\text{Val}_{y_i}(G_{f_m}) < m_{y_i} + \varepsilon/2.$$  

Now for any fixed strategy of player $A$, let $B$ play according to the assumption, to make $E(g) < m_x - \varepsilon$ and then play $\varepsilon/2$ optimally in $G_{f_m}$ to make

$$E_x(f_m) \leq \sum_{i=1}^k p(y_i) \text{Val}_{y_i}(G_{f_{n_i}}) + \varepsilon/2 < \sum_{i=1}^k p(y_i) m_{y_i} + \varepsilon$$

(by (1))

$$= E(g) + \varepsilon < m_x$$

(by assumption) which contradicts the fact that $m_x = \lim_n \text{Val}_x(G_{f_n})$, and the lemma is proved. \(\square\)

Now we are ready for the

**Proof of Theorem 2.1.** We shall show that for fixed $\varepsilon > 0$, $A$ can guarantee that $E(f) \geq \lim_n \text{Val}(G_{f_n}) - \varepsilon = m_x - \varepsilon$ (where $\varepsilon$ denotes the empty sequence). This will complete the proof, since $\{\text{Val}(G_{f_n})\}$ is a non-increasing sequence, and so $U(G_f) \geq \lim \text{Val}(G_{f_n})$.

First, let $A$ play optimally in $G^*_x$, and then, if $x_n$ is the position after the $n$th move, let $A$ play optimally in $G^*_{x_n}$. Define the random variables $X_0 = m_x$; if $n \geq 1$, $X_n = m_{x_n}$ if $x_n$ is the position after the first $n$ moves. By the lemma, we have $E(X_n | X_{n-1} \cdots X_0) \geq X_{n-1} \Rightarrow$

$$E(X_n) \geq m_x$$

for all $n$.

Now, using the usual facts about upper semicontinuity, for fixed $k$ (if $z = (z_1, z_2, \ldots)$ is the resulting sequence of moves), there exists
N(\theta, z, e) such that if \( n \geq N(\theta, z, e) \), any sequence \( \omega = (\omega_1, \omega_2, \cdots) \) agreeing with \( z \) up to the \( n \)th move has the property

\[
 f_k(\omega) < f_k(z) + \varepsilon \Rightarrow \text{Val}(G_{f_k}) < f_k(z) + \varepsilon
\]

\[
 m_{(z_1, z_2, \ldots, z_n)} < f_k(z) + \varepsilon
\]

\[
 \Rightarrow \text{for all } z, \text{ } \limsup X_n(z) < f_k(z) + \varepsilon
\]

\[
 \Rightarrow (\text{by Fatou}) \text{ } \limsup E(X_n) < E(f_k) + \varepsilon
\]

\[
 \Rightarrow E(f_k) > m_e - \varepsilon \text{ for all } k
\]

(by the dominated convergence theorem). □

**Corollary 1.** If \( f_n \) are lower semicontinuous, \( f \) then \( \lim_n \text{Val}(G_{f_n}) = \text{Val}(G_f) \).

**Proof.** The negative of an u.s.c. function is l.s.c. so the theorem applies by reversing the roles of the players.

**Corollary 2.** Games with lower semicontinuous payoff functions can be approximated by finite games.

**Proof.** Suppose \( f \) is l.s.c. Define \( f_n \) by \( f_n(\nu) = \inf_{\omega \in S} f(\omega) \) where \( S = \{ \omega \in \Omega | \text{1st } n \text{ coordinates of } \omega \text{ agree with the 1st } n \text{ coordinates of } \nu \} \). Then the games \( G_{f_n} \) are “finite”, since the payoff is decided in the first \( n \) moves. But the fact that \( f \) is l.s.c. implies \( f_n \uparrow f \), so we just apply Corollary 1. (The functions \( f_n \) are continuous.)

**Corollary 3.** Open games can be approximated by finite games, i.e., if \( f = I_\varnothing \) where \( \varnothing \) is an open set (in the product topology on \( \Omega \)) then the game \( G \) can be approximated by the games \( G_n \), where the payoff in \( G_n \) is 1 if \( \varnothing \) is hit by the \( n \)th move, 0 otherwise. (This is actually a special case of Corollary 2.)

**Proof.** Immediate since \( I_\varnothing \) is l.s.c.

A **Counterexample.** Approximation theorems do not exist in general as the following example shows. Let \( A = B = \{0, 1\} \) for \( n = 1, 2, \cdots \), so \( \Omega = \prod_{n=1}^{\infty} \{0, 1\} \times \{0, 1\} \). Let \( S_n = F_n \cup G \) where \( F_n = \{ \omega \in \Omega | \exists i \leq n \text{ with } \omega_i = (1, 1) \} \) (in other words \( F_n = \{ \omega | \text{both players say } 1 \text{ on the same move sometime before the } n \text{th move} \})\), and \( G = \{ \omega \in \Omega | \text{player } B \text{ says } 0 \text{ on every move} \} \). Clearly \( F_n \) and \( G \) are closed sets, so the functions \( I_{S_n} \) are upper semicontinuous. Now the games \( G_n \) with payoffs \( I_{S_n} \) have value 0 since player \( B \) need only say 0 for the first \( n \) moves and 1 sometime after that to keep play from hitting \( S_n \). Also since \( S_{n+1} = S_n \) for all \( n \), \( I_{S_n} \uparrow I_G \) where \( S = \bigcup_{n=1}^{\infty} S_n \). But the game with payoff \( I_G \) has value 1 which player \( A \)
can achieve by merely saying 1 on every move. Player B either must say 0 every time or 1 sometime and so $S$ is hit.

**AN OPEN QUESTION.** We do not know whether if $f_n$ are continuous, $f_n \to f$, then $\text{Val } G(F_n) \to \text{Val } G(F)$. This question has some relevance to the study of stochastic games (see [2], [3]).

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**REFERENCES**