THE COMPLETION OF A RING WITH A VALUATION

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Abstract. This paper proves three main results: the completion of a commutative ring with respect to a Manis valuation is an integral domain; a necessary and sufficient condition is given that the completion be a field; and the completion is a field when the valuation is Harrison and the value group is archimedean ordered.

1. Introduction. In [1] we defined a generalized pseudovaluation on a ring \( R \) and found explicitly the completion of \( R \) with respect to the topology induced on \( R \) by such a pseudovaluation. Using this inverse limit characterization of the completion of \( R \) we show, in §2, that when the codomain of a valuation on \( R \) is totally ordered, the completion of \( R \) with respect to the valuation has no (nonzero) divisors of zero and the valuation on \( R \) can be extended to a valuation on the completion. In §3, \( R \) is assumed to have an identity and so, from [1, §2], the completion has an identity: a necessary and sufficient condition is given such that each element of the completion of \( R \) has a right inverse.

§§2 and 3 are immediately applicable to a Manis valuation \( \varphi \) on a commutative ring \( R \) with identity [3], where the valuation \( \varphi \) is surjective and the set \( \varphi(R) \) is a totally ordered group. A Harrison valuation on \( R \) is a Manis valuation \( \varphi \) on \( R \) such that the set \( \{ x : x \in R, \varphi(x) > \varphi(1) \} \) is a finite Harrison prime [2]. In §4 we show that the completion of a ring with respect to a Harrison valuation is a field if the value group is archimedean ordered.

Let \( R \) be a ring (not necessarily either commutative or with identity). Let \( S \) be a totally ordered quasi-residuated [1, §1] semigroup and let \( S_0 \) be the disjoint union of \( S \) and a zero element 0\(_S\) with the properties: 0\(_S\)0\(_S\) = 0\(_S\); and, for any \( s \in S \), 0\(_S\) > s and s 0\(_S\) = 0\(_S\) = 0\(_S\) s. Let \( \varphi : R \to S_0 \) be a valuation on \( R \). That is, for all \( a, b \in R \),

\[
\begin{align*}
(1.1) \quad \varphi(ab) & = \varphi(a)\varphi(b); \\
(1.2) \quad \varphi(a-b) & \geq \min\{\varphi(a), \varphi(b)\}; \\
(1.3) \quad \varphi(0) & = 0_S; \\
(1.4) \quad \text{the set } \varphi(R) \setminus \{0_S\} \text{ is nonempty.}
\end{align*}
\]

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Note that, if \( \varphi(a) \neq \varphi(b) \), then
\[
\varphi(a - b) = \min\{\varphi(a), \varphi(b)\}.
\]

Now if \( S \) has a greatest element \( \pi \), it is clear from [1] that the completion of \( R \) with respect to \( \varphi \) is isomorphic to the ring \( R/\{x : x \in R, \varphi(x) \geq \pi\} \). Hence let \( S \) have no greatest element.

Following [1], let \( \{P_s\}_{s \in S} \) be the generalized filtration defined on \( R \) as follows: For each \( s \in S \), \( P_s = \{x : x \in R, \varphi(x) \geq s\} \). Note that, in this case, \( \bigcup_{s \in S} P_s = R \). From [1, §2], the completion of \( R \) with respect to \( \varphi \) is the topological ring \( R = \text{proj lim } R/P_t \), with zero \( 0 = \{P_s\}_{s \in S} \), and addition, multiplication and topology defined in [1, §2]. For reference, \( R = \{\{\xi_s\}_{s \in S} \in \prod_{s \in S} R/P_s : \text{for all } s, t \in S \text{ such that } s \leq t, \xi_s \leq \xi_t\} \). When there is no risk of ambiguity, \( \{\xi_s\}_{s \in S} \) will be written as \( \{\xi_s\} \).

2. \( R \) has no divisors of zero. First we need

**Lemma 1.** Let \( \{\xi_s\}_{s \in S} \in R \setminus \{0\} \). Then there exist \( u, v \in S \) with \( u < v \) such that \( \xi_u = P_u, \xi_v \neq P_v \), and, for all \( x, y \in \xi_u \), \( u \leq \varphi(x) = \varphi(y) < v \).

**Proof.** (i) Let \( s \in S, x \in \xi_s \). Since \( \bigcup_{s \in S} P_s = R \), there exists \( t \in S \) such that \( x \in P_t \). If \( t \geq s \), then \( x \in P_s \) from [1, (1.2)], and so \( \xi_s = P_s \). Otherwise \( t < s \), \( \xi_s \subseteq \xi_t \), and so \( \xi_s = x + P_t = P_t \). Hence there exists \( u \in S \) such that \( \xi_u = P_u \).

(ii) Since \( \{\xi_s\}_{s \in S} \neq 0 \), there exists \( v \in S \) such that \( \xi_v \neq P_v \). Now if \( v \leq u \), then \( P_u = \xi_u \subseteq \xi_v \), and so \( \xi_v = P_v \). Therefore \( v > u \).

(iii) Now \( \varphi(x) < v \) for all \( x \in \xi_v \), since otherwise \( \xi_v = P_v \). Since \( v > u \), \( \xi_v \subseteq \xi_u = P_u \) and so \( \varphi(x) \geq u \) for all \( x \in \xi_v \). Suppose there exist \( y, z \in \xi_v \) such that \( \varphi(y) \neq \varphi(z) \). Then \( \min\{\varphi(y), \varphi(z)\} = \varphi(y - z) \geq v \) since \( y - z \in P_v \), which contradicts \( \varphi(y), \varphi(z) < v \). Hence \( \varphi(x) = \varphi(y) \) for all \( x, y \in \xi_v \). This completes the proof.

**Theorem 1.** \( R \) has no divisors of zero.

**Proof.** Let \( \{\xi_s\}, \{\eta_s\} \in R \) such that \( \{\xi_s\}, \{\eta_s\} \neq 0 \). Suppose that \( \{\xi_s\}, \{\eta_s\} \neq 0 \). Then, by Lemma 1, there exist \( a, b, t, u \in S \) such that, for all \( x \in \xi_t \), \( \varphi(x) = a \); and, for all \( y \in \eta_u \), \( \varphi(y) = b \).

Since \( S \) has no maximal element, there exists \( c \in S \) such that \( c > ab \); and, by the definition of multiplication in \( R \) [1, §2], there exists \( v \in S \) such that, for all \( x \in \xi_v \) and for all \( y \in \eta_v \), \( xy \in P_c \).

Let \( w \leq t, u, v \). Then, for all \( x \in \xi_w \subseteq \xi_t \), and for all \( y \in \eta_w \subseteq \eta_u \), \( \varphi(xy) = ab < c \), which contradicts \( xy \in P_c \).

Hence \( \{\xi_s\} = 0 \) or \( \{\eta_s\} = 0 \) which completes the proof.
Corollary. The completion of a commutative ring with respect to a nontrivial Manis valuation is an integral domain.

Lemma 1 also enables us to define a valuation on $R$. Let $\{\xi_s\} \in R \setminus \{0\}$. Then there exists $t \in S$ such that $\xi_t \neq P_t$. Define $\varphi(\xi_s) = \varphi(x)$ where $x \in \xi_t$. Define $\varphi(0) = 0_S$. Then it is a straightforward task to show that $\varphi : R \to S_0$ is a well defined valuation on $R$. Moreover, the valuation $\varphi$ on $R$ is an "extension" of the valuation $\varphi$ on $R$ in the sense that $\varphi$ restricted to the image $i(R)$ of $R$ in $R$, defined in [1, Theorem 2.1], satisfies $\varphi(i(x)) = \varphi(x)$ for each $x \in R$. And it is easy to show that $\varphi$ defines the topology $\mathcal{T}$ [1, §2] on $R$.

3. When elements of the completion have inverses. Let $R$ have identity 1. Then, from [1, Proposition 2.1], $R$ has identity $\{1 + P_s\}_{s \in S}$.

Proposition 1. Each element of $R$ has a right inverse if and only if (1) given $a \in R$ such that $\varphi(a) \neq 0_S$, and given $s \in S$, there exists $b \in R$ such that $\varphi(ab - 1) \geq s$.

Proof. (i) Let each element of $R$ have a right inverse. Let $a \in R$ be such that $\varphi(a) \neq 0_S$, and let $w \in S$. Clearly, $\{a + P_s\} \neq 0$ since $S$ has no maximal element. Hence there exists $\{\eta_s\} \in R$ such that $\{a + P_s\}\{\eta_s\} = \{1 + P_w\}$. By the definition of multiplication in $R$, there exists $t \in S$ such that, for all $x \in \eta_t$, $ax \in 1 + P_w$. Hence (1) holds.

(ii) Conversely, let (1) hold. Denote $\varphi(1)$ by $1_S$. Let $\{\xi_s\} \in R \setminus \{0\}$. By Lemma 1 there exist $\rho, \sigma \in S$ such that $\xi_\rho \neq P_\rho$ and, for all $x \in \xi_\rho$, $\varphi(x) = \sigma$; $\sigma$ is a constant for the chosen $\{\xi_s\}$.

For each $s \in S$, let $x_s \in \xi_s$ and let $y_s \in R$ be such that $\varphi(x_s y_s - 1) > \sigma_S$ ($S$ has no maximal element); let $m_s \in S$ be such that $m_s > \sigma_S$ and let $g(s) \in S$ be such that $g(s) \geq m_s$, $s$. Let $t \in S$ be such that $\sigma > 1_S$ and $t \geq \rho$. Then, for each $s \in S$ such that $s \geq t$, $\sigma \varphi(y_s) = \varphi(x_s y_s) = 1_S$. Define, for each $s \in S$,

$$\eta_s = y_{\varphi(s)} + P_s \quad \text{if } s \geq t,$$

$$= y_{\varphi(t)} + P_s \quad \text{if } s < t.$$

We shall show that $\{\xi_s\}\{\eta_s\} = \{1 + P_s\}$.

(a) We must show that, for each $s \in S$, $\eta_s$ is well defined. Accordingly, for each $s \in S$, let $w_s \in \xi_s$, $z_s \in R$ and $\tau, u, n_s, h(s) \in S$ be such that $\xi_t \neq P_t$; $\varphi(w_s z_s - 1) > \sigma_S$; $n_s > \sigma_S$ and $h(s) \geq n_s$, $s$; $\sigma u > 1_S$ and $u \geq \tau$; and

$$\xi_s = z_{h(s)} + P_s \quad \text{if } s \geq u,$$

$$= z_{h(u)} + P_s \quad \text{if } s < u.$$

We must show that, for each $s \in S$, $\xi_s = \eta_s$.

Claim. Let $a, b, c \in S$ be such that $a, b \geq c \geq t$. Then $z_{h(a)} - y_{\varphi(b)} \in P_c$. 

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SUBPROOF. We shall refer to $h(a)$ and $g(b)$ as $h$ and $g$ respectively. Now

$$w_h(z_h - y_g) = (w_h z_h - x_g y_g) + (x_g - w_h) y_g.$$ 

Clearly, $\varphi(w_h z_h - 1 - (x_g y_g - 1)) > ac$. Also $\varphi(x_g - w_h) \geq r \sigma$ where $r = \min\{m_5, \eta_a\} > ac$. Hence $\varphi(x_g - w_h) y_g > ac$ since $\sigma \varphi(y_g) = 1_s$, and so $\varphi(w_h z_h - y_g) > ac$. Thus $\varphi(z_h - y_g) \geq c$ since $\varphi(w_h) = \sigma$. This proves the claim.

Let $s \in S$. Without loss of generality, let $u \geq t$. Then, by applying the claim to each of the cases $s \geq u$, $u \geq s \geq t$, $t > s$, we see that $\zeta_s = \eta_s$. Thus $\eta_s$ is well defined for each $s \in S$.

(b) By further application of the claim, it is easy to show that $\{\eta_s\} s \in S \in R$.

(c) Finally we must show that $\{\xi_s\} \{\eta_s\} = \{1 + P_s\}$. Let $\{\xi_s\} \{\eta_s\} = \{\Omega_s\}$. Let $s \in S$. Then there exists $v \in S$ such that $xy \in \Omega_s$ for all $x \in \xi_v$, $y \in \eta_v$. Let $d \in S$ be such that $ad \geq s$ and $d \geq v$, $t$. Then $x_{\varphi(d)} \in \xi_v$, $y_{\varphi(d)} \in \eta_v$ and $\varphi(x_{\varphi(d)} y_{\varphi(d)} - 1) \geq s$. Hence $\Omega_s = x_{\varphi(d)} y_{\varphi(d)} + P_s = 1 + P_s$, which completes the proof of the proposition.

COROLLARY. Let $\varphi : R \rightarrow S_0$ be a nontrivial Manis valuation on a commutative ring $R$ with identity $1$. Then the completion of $R$ with respect to $\varphi$ is a field if and only if condition (1) holds.

4. Harrison valuations. In [2, pp. 11 and 12 and Proposition 2.2, p. 14], Harrison shows that, given a finite prime $P$ of a commutative ring $R$ with identity, one can define a Manis valuation $\varphi_P : R \rightarrow \Gamma_P$ on $R$ such that the set $\{x : x \in R, \varphi_P(x) > \varphi_P(1)\} = P$. Using the corollary to Proposition 1, we have:

THEOREM 2. Let $\varphi : R \rightarrow S_0$ be a nontrivial Harrison valuation on a commutative ring $R$ with identity $1_R$, such that $S$ be archimedean ordered. Then the completion of $R$ with respect to $\varphi$ is a field.

PROOF. Let $a \in R$ be such that $\varphi(a) \neq 0_S$ and let $s \in S$. Now $\{x : x \in R, \varphi(x) > 1_s\} = P$, a finite Harrison prime of $R$. Suppose $aR \subseteq P$. Then, for all $x \in R$, $\varphi(a) \varphi(x) > 1_s$, which contradicts $\varphi(a) \neq 0_S$ since $\varphi$ is surjective and $S$ is a totally ordered group. Hence $aR + P$ properly contains $P$. But $aR + P$ is closed under addition and multiplication. Thus $-1_R \in aR + P$, since $P$ is a Harrison prime: that is, there exist $c \in R$, $d \in P$ such that $ac - 1_R = d$.

Since $\varphi(d) > 1_s$ and $S$ is archimedean ordered, there exists a natural number $m$ such that $\varphi(d^m) = (\varphi(d))^m \geq s$. Let $n$ be an odd integer such that $n \geq m$. Then $\varphi(d^n) \geq s$. Now $d^n = (ac - 1_R)^n = ab - 1_R$ where $b \in R$. The theorem then follows from Proposition 1.
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REFERENCES


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