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UNIQUENESS OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR SINGULAR ULTRAHYPERBOLIC EQUATIONS

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ABSTRACT. The paper gives necessary and sufficient conditions for uniqueness of solutions of the Dirichlet problem for ultrahyperbolic partial differential equations with multiple singular lines.

1. **Introduction.** In [1], Diaz and Young obtained necessary and sufficient conditions for the uniqueness of solutions of certain improperly posed problems for the ultrahyperbolic equation

$$\Delta u - \sum_{j,k=1}^{n} (a_{jk}u_{y_j})_{y_k} + cu = 0 \tag{1}$$

where the symbol Δ denotes the Laplace operator in the variables $x_1, \cdots, x_m$. Their results have been extended recently in [2] to the singular ultrahyperbolic equation

$$u_{tt} + \frac{\alpha}{t} + \Delta u - \sum_{j,k=1}^{n} (a_{jk}u_{y_j})_{y_k} + cu = 0 \tag{2}$$

where α is a real parameter, $-\infty<\alpha<\infty$.

The purpose of this paper is to present some results on the uniqueness of the Dirichlet problem for the class of ultrahyperbolic equations

$$Lu \equiv \sum_{i=1}^{m} \left(u_{x_i x_i} + \frac{\alpha_i}{x_i} u_{x_i}\right) - \sum_{j,k=1}^{n} (a_{jk}u_{y_j})_{y_k} + cu = 0 \tag{3}$$

with multiple singular lines and real parameters α_i, $-\infty<\alpha_i<\infty$.

As usual we consider the boundary value problem in the domain $Q = X\times Y$, where X is a parallelepiped defined by $0<x_i<a_i$ $(1\leqq i\leqq m)$ and Y

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is a bounded domain in the space $y_1, \cdots, y_n$. We assume that the coefficients a_{jk} and c are functions of the variables $y_1, \cdots, y_n$ alone with $c \geqq 0$ in Y. Further, we assume that the matrix (a_{jk}) is symmetric and positive definite, and that a_{jk}, c and the boundary ∂Y of Y are smooth enough to allow the application of the divergence theorem and the existence of a complete set of eigenfunctions for the eigenvalue problem that we will need below.

2. **A lemma.** An interesting feature of the differential equation (3) is that it possesses several singular lines. As a result of this, every solution of the equation which is smooth on the singular lines satisfies a regularity condition on those lines. This behavior was first observed by Walter [3] in the case of the normal hyperbolic Euler-Poisson-Darboux equation, and later by Fox [4] for a slightly general singular equation. We shall prove this property for our equation (3) as a lemma.

LEMMA. *If $u \in C^2(Q) \cap C^1(\bar{Q})$ satisfies $Lu=0$, then $u_{x_i}=0$ on $x_i=0$ for every index i $(1 \leqq i \leqq m)$ for which $\alpha_i \neq 0$.*

PROOF. Let us assume that $\alpha_m \neq 0$ and set $x'=(x_1, \cdots, x_{m-1})$ and $z=x_m$. Denote by X^* the parallelepiped defined by $0<\varepsilon_i<x_i<a_i$ $(1 \leqq i \leqq m-1)$, $0<z<t$ $(t<a_m)$, and by X' the resulting parallelepiped when the variable z is deleted. We integrate over $Q^*=X^* \times Y$ the identity

$$\begin{aligned} 2z^\beta u_z Lu = {} & [z^\beta(u_z^2 - u_{xi}^2 + a_{jk}u_{y_j}u_{y_k} + cu^2)]_z \\ & + (2z^\beta u_z u_{xi})_{xi} - (2z^\beta a_{jk} u_z u_{y_j})_{y_k} \\ & - \beta z^{\beta-1}(u_z^2 - u_{xi}^2 + a_{jk}u_{y_j}u_{y_k} + cu^2) \\ & + 2z^\beta u_z \alpha_i u_{x_i}/x_i + 2\alpha_m z^{\beta-1}u_z^2 \end{aligned}$$

where $\beta>0$. Here, as well as in the rest of our discussion, we adopt the convention of summing over repeated indices. Applying the divergence theorem and noting that u is of class C^1 in $\bar{Q}$, we obtain

$$\begin{aligned} & t^\beta \int_{Q'} F(u)\,dx'\,dy + \int_{\partial Q^*} [2z^\beta u_z(u_{x_i}\nu_{x_i} - a_{jk}u_{y_j}\nu_{y_k})]\,dS \\ (4) \qquad & - \int_{Q^*} \beta z^{\beta-1}[F(u) - 2zu_z\alpha_i u_{x_i}/(\beta x_i) - 2\alpha_m u_z^2/\beta]\,dx'\,dy\,dz = 0 \end{aligned}$$

where

$$F(u) = u_z^2 - u_{xi}^2 + a_{jk}u_{y_j}u_{yk} + cu^2,$$

$Q'=X' \times Y$ and (ν_{x_i}, ν_{y_k}) is the outward unit normal vector on $\partial Q'$.

Let us divide (4) throughout by t^β and take the limit as t approaches zero. We observe that the second integral drops out in the limit as its integrand is bounded and $(z/t)<1$. On the other hand, the third integral becomes a derivative with respect to z^β at $z=0$ as may be seen by writing it in the form

$$\frac{1}{t^\beta}\int_0^{t^\beta}\int_{Q'}[F(u)-2zu_z\alpha_iu_{x_i}/(\beta x_i)-2\alpha_m u_z^2/\beta]\,dx'\,dy\,d(z^\beta).$$

Thus, in the limit, we obtain from (4)

$$(2\alpha_m/\beta)\int_{Q'}u_z^2(x',0,y)\,dx'\,dy=0.$$

Since $\alpha_m\neq 0$, this implies that $u_z(x',0,y)=0$ in Q'. By continuity, this holds for $0\leqq x_i\leqq a_i$ $(1\leqq i\leqq m-1)$ and for all y in $\bar{Y}$.

3. **The Dirichlet problem.** Let us now consider the homogeneous Dirichlet problem

$$(5)\qquad Lu=0\quad\text{in } Q,\qquad u=0\quad\text{on }\partial Q.$$

By changing subscripts if necessary, we can assume that α_i, $i=1,\cdots,p$, are not necessarily zero and $\alpha_j=0$, $j=p+1,\cdots,m$ with $0\leqq p\leqq m$.

THEOREM 1. *If $\alpha_i>0$ for all i $(1\leqq i\leqq p)$, then every solution $u\in C^2(Q)\cap C^1(\bar{Q})$ of the problem* (5) *vanishes identically in Q.*

PROOF. Suppose that $u\in C^2(Q)\cap C^1(\bar{Q})$ is a solution of (5). By the hypothesis, it follows from the lemma that $u_{x_i}=0$ on $x_i=0$ for $i=1,\cdots,p$. Now let us integrate over Q the identity

$$\begin{aligned}2u_{x_r}Lu&\equiv(2u_{x_r}u_{x_i})_{x_i}-(u_{x_i}^2-a_{jk}u_{y_j}u_{y_k}-cu^2)_{x_r}\\&\quad-(2a_{jk}u_{x_r}u_{y_j})_{y_k}+2u_{x_r}\alpha_iu_{x_i}/x_i\\&=0\end{aligned}$$

$(1\leqq r\leqq p)$ and use the divergence theorem. Because $u=0$ on ∂Q and $u_{x_i}=0$ on $x_i=0$ for $i=1,\cdots,p$, we obtain

$$(6)\qquad\int_{X_r\times Y}u_{x_r}^2(a_r,x',y)\,dx'\,dy+2\int_Q u_{x_r}\alpha_iu_{x_i}/x_i\,dx\,dy=0$$

where X_r denotes the parallelepiped defined by $0<x_i<a_i$, $1\leqq i\leqq m$, $i\neq r$, and x' denotes a point in X_r. Taking the sum of (6) with respect to r leads then to the equation

$$(7)\quad\sum_{r=1}^{p}\int_{X_r\times Y}u_{x_r}^2(a_r,x',y)\,dx'\,dy+2\int_Q\left(\sum_{r=1}^{p}u_{x_r}\right)\left(\sum_{i=1}^{p}\alpha_iu_{x_i}/x_i\right)dx\,dy=0$$

where we observe that each of the summands in the first sum is nonnegative.

Now, by the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{p} u_{x_i}\right)\left(\sum_{i=1}^{p} \alpha_i u_{x_i}/x_i\right) \geqq \left(\sum_{i=1}^{p} (\alpha_i/x_i)^{1/2} u_{x_i}\right)^2 \geqq 0. \tag{8}$$

Therefore, the integrand of the second integral in (7) is also nonnegative. Hence we conclude, from (7) and (8), that

$$\sum_{i=1}^{p} (\alpha_i/x_i)^{1/2} u_{x_i} = 0. \tag{9}$$

If we multiply (9) by $2u$ and integrate the result over the domain D defined by $0<x_i<a_i$ $(1\leqq i\leqq p)$, we finally obtain

$$\sum_{i=1}^{p} \int_D (\alpha_i^{1/2} u^2/x_i^{3/2})\, dx_1 \cdots dx_p = 0$$

since $u=0$ on ∂D. This yields the conclusion of the theorem.

When the parameters α_i are all nonpositive, we have the following result.

THEOREM 2. *Let λ_k $(k=1, 2, \cdots)$ be the eigenvalues of the problem*

$$\begin{aligned} (a_{jk}v_{y_j})_{y_k} - cv + \lambda v &= 0 \quad \text{in } Y, \\ v &= 0 \quad \text{on } \partial Y. \end{aligned} \tag{10}$$

If $\alpha_i \leqq 0$ for $i=1, \cdots, p$, then every solution $u \in C^2(Q)\cap C^1(\bar{Q})$ of the problem (5) vanishes identically in Q if, and only if, for all nonzero real numbers $\mu_1, \cdots, \mu_p$ and for all nonzero integers $b_{p+1}, \cdots, b_m$ satisfying the condition

$$\sum_{i=1}^{p} \mu_i + \sum_{i=p+1}^{m} (b_i \pi x_i/a_i)^2 = \lambda_k \tag{11}$$

there exists μ_r $(1\leqq r\leqq p)$ such that

$$J_{(1-\alpha_r)/2}(\mu_r^{1/2} a_r) \neq 0 \tag{12}$$

where $J_p(t)$ denotes the Bessel function of the first kind of order p.

PROOF. The necessity part follows easily. Let λ_s be an eigenvalue of (10) and v_s a corresponding eigenfunction. Suppose that there exist nonzero real numbers $\nu_1, \cdots, \nu_p$ and nonzero integers $q_{p+1}, \cdots, q_m$ satisfying (11) such that

$$J_{(1-\alpha_i)/2}(\nu_i^{1/2} a_i) = 0, \qquad i = 1, \cdots, p.$$

Then the function

$$u(x, y) = \prod_{i=1}^{p} x_i^{(1-\alpha_i)/2} J_{(1-\alpha_i)/2}(\nu_i^{1/2} x_i)\phi(x; q) v_s(y)$$

where

$$\phi(x;q) = \prod_{i=p+1}^{m} \sin(q_i \pi x_i / a_i) \tag{13}$$

is a nontrivial solution of the problem (5) as is readily verified.

Conversely, suppose that condition (12) holds. Let λ_k be an eigenvalue of (10) and let μ_i be any nonzero real numbers such that

$$J_{(1-\alpha_i)/2}(\mu_i^{1/2} a_i) = 0 \tag{14}$$

for all i $(1 \leqq i \leqq p)$ except for $i=r$. For any nonzero integers $b_{p+1}, \cdots, b_m$, let μ_r be determined by the relation (11). Notice that μ_r may very well be a negative number and that from the hypothesis we have

$$J_{(1-\alpha_r)/2}(\mu_r^{1/2} a_r) \neq 0. \tag{15}$$

Now let

$$w(x, y) = \prod_{i=1}^{p} x_i^{(1+\alpha_i)/2} J_{(1-\alpha_i)/2}(\mu_i^{1/2} x_i) \phi(x; b) v_k(y) \tag{16}$$

where $\phi(x; b)$ is defined in (13) and v_k is an eigenfunction of (10) corresponding to λ_k. By direct differentiation, making use of (10) and (11), it is readily shown that (16) satisfies the adjoint equation

$$Mw \equiv w_{x_i x_i} - \alpha_i (w/x_i)_{x_i} - (a_{jk} w_{y_j})_{y_k} + cw = 0$$

of (3). Moreover, from (13), (14), and (15), it is clear that (16) vanishes on ∂Q except on $x_r = a_r$.

Consider the Green's formula

$$\begin{aligned} \int_{Q_s} &(wLu - uMw)\, dx\, dy \\ &= \int_{\partial Q_s} [(wu_{x_i} - uw_{x_i} + (\alpha_i/x_i) uw) \nu_{x_i} - a_{jk}(wu_{y_j} - uw_{y_j}) \nu_{y_k}]\, dS \end{aligned} \tag{17}$$

where $Q_s = X_s \times Y$, X_s being the parallelepiped defined by $0 < s_i < x_i < a_i$ for $i = 1, \cdots, p$, and $0 < x_i < a_i$ for $i = p+1, \cdots, m$, and (ν_{x_i}, ν_{y_k}) is the outward unit normal vector on ∂Q_s. If u is a solution of (5) and w is chosen as the function (16), then substitution of these functions in (17) yields

$$\int_{\partial Q_s} (wu_{x_i} - uw_{x_i} + (\alpha_i/x_i) uw) \nu_{x_i}\, dS = 0. \tag{18}$$

Now let s_i $(1 \leqq i \leqq p)$ approach zero. Since w_{x_i} and w/x_i are bounded at $x_i = 0$ for $i = 1, \cdots, p$, and $u = 0$ on $x_i = 0$, $x_i = a_i$ for all i, $1 \leqq i \leqq m$, it follows from (13), (14), and (16) that (18) reduces to

$$(19)\quad a_r^{(1+\alpha_r)/2}J_{(1-\alpha_r)/2}(\mu_r^{1/2}a_r)\cdot\int u_{x_r}(a_r, x', y)\psi(x;\mu)\phi(x;b)v_k(y)\,dx'\,dy = 0$$

where

$$\psi(x;\mu) = \prod_{i=1;i\neq r}^{p} x_i^{(1+\alpha_i)/2}J_{(1-\alpha_i)/2}(\mu_i^{1/2}x_i).$$

X' is the subspace of X with x_r deleted and x' denotes a point in X'. In view of (15), we conclude that

$$(20)\quad \int_{X'\times Y} u_{x_r}(a_r, x', y)\psi(x;\mu)\phi(x;b)v_k(y)\,dx'\,dy = 0$$

for all nonzero μ_i numbers satisfying (14), for all nonzero integers $b_{p+1}, \cdots, b_m$, and for all eigenfunctions v_k. By the completeness of the sets of eigenfunctions $\{\psi(x;\mu)\}$, $\{\phi(x;b)\}$, and $\{v_k\}$ in the respective spaces $\{x_i|0<x_i<a_i, 1\leqq i\leqq p, i\neq r\}$, $\{x_i|0<x_i<a_i, p+1\leqq i\leqq m\}$, and Y, we conclude that

$$(21)\quad u_{x_r} = 0 \quad \text{on } x_r = a_r.$$

Finally, by integrating the identity

$$\begin{aligned}(2x_ru_{x_r} + u)Lu = {} & [x_r(-u_{xi}^2 + a_{jk}u_{y_j}u_{y_k} + cu^2)]_{x_r}\\ & + [2x_ru_{x_r}u_{xi} + uu_{xi} + (\alpha_iu^2)/(2x_i)]_{x_i}\\ & - [a_{jk}(2x_ru_{x_r} + u)u_{y_j}]_{y_k}\\ & - (2u_{xr}^2 - (\alpha_iu^2)/(2x_i^2) - 2x_ru_{x_r}(\alpha_iu_{x_i}/x_i)\end{aligned}$$

over Q and using (21), we obtain

$$(22)\quad \int_Q\left(2u_{x_r}^2 - \frac{u^2\alpha_i}{2x_i^2} - 2x_ru_{x_r}\frac{\alpha_i}{x_i}u_{x_i}\right)dx\,dy = 0.$$

Since r is arbitrary, summing (22) with respect to r from 1 to p yields

$$(23)\quad \int_Q\left[2\sum u_{xi}^2 - \frac{pu^2}{2}\sum(\alpha_i/x_i^2) - 2\sum(x_iu_{x_i})\sum(\alpha_iu_{x_i}/x_i)\right]dx\,dy = 0$$

where we have dropped the limits of summation for convenience. Since $\alpha_i\leqq 0$, $1\leqq i\leqq p$, we see by the Cauchy-Schwarz inequality that

$$\left(\sum(-\alpha_i)^{1/2}u_{x_i}\right)^2 \leqq -\sum(x_iu_{x_i})\sum(\alpha_iu_{x_i}/x_i)$$

so that $\sum(x_iu_{x_i})\sum(\alpha_iu_{x_i}/x_i)\leqq 0$. It follows that the integrand in (23) is nonnegative from which we conclude that u vanishes identically in Q. This completes the proof of the theorem.

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