ON THE INDIVIDUAL ERGODIC THEOREM FOR POSITIVE OPERATORS

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Abstract. A theorem which gives a condition on a positive linear contraction on an $L^1$-space in order that the individual ergodic theorem hold is proved. The theorem contains a result obtained by Y. Ito as a special case.

Let $(X, \mathcal{M}, m)$ be a $\sigma$-finite measure space and let $T$ be a positive linear contraction on $L^1(m)$. Let $a_{n,j}$ be a matrix of numbers such that

\begin{align*}
(1) & \quad \sum_{j=0}^{\infty} |a_{n,j}| < \infty \quad \text{for } n = 0, 1, \ldots, \\
(2) & \quad \lim_{n' \to \infty} \sum_{j=0}^{\infty} a_{n',j} = 1, \\
(3) & \quad \lim_{n' \to \infty} \sum_{j=0}^{\infty} a_{n',j} b_{j+1} = b
\end{align*}

whenever $b_0, b_1, \ldots$ is a bounded sequence of numbers for which $\lim_{n'} \sum_{j=0}^{\infty} a_{n',j} b_j = b$ exists and is finite, where $\{n'\}$ is a subsequence of $\{n\}$.

Under these conditions we shall prove the following

**Theorem.** If there exists a strictly positive function $h$ in $L^1(m)$ such that the set $\{\sum_{j=0}^{\infty} a_{n,j} T^j h; n \geq 0\}$ is weakly sequentially compact in $L^1(m)$, then for any $f \in L^1(m)$ the limit

\[ \lim_{n} \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) \]

exists and is finite almost everywhere.

The following proof is a generalization of that given by Y. Ito in [6].

**Proof.** Let $g \in L^1(m)$ and $\{n'\}$ a subsequence of $\{n\}$ such that

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\[ \sum_{j=0}^{\infty} a_{n',j} T^j h \text{ converges weakly to } g. \] Then for any \( u \in L^\infty(m) \) we have
\[
\int gu \, dm = \lim_{n'} \sum_{j=0}^{\infty} a_{n',j} \int (T^j h)u \, dm
\]
\[ = \lim_{n'} \sum_{j=0}^{\infty} a_{n',j} \int (T^{j+1} h)u \, dm = \int (Tg)u \, dm. \]
This implies that \( g = Tg \). Next suppose that \( \int (f - Tf)u \, dm = 0 \) for any \( f \in L^1(m) \). Then, clearly, \( \int (f - T^nf)u \, dm = 0 \) for all \( n \geq 0 \), and hence
\[
\int gu \, dm = \lim_{n'} \sum_{j=0}^{\infty} a_{n',j} \int (T^j h)u \, dm
\]
\[ = \lim_{n'} \sum_{j=0}^{\infty} a_{n',j} \int hu \, dm = \int hu \, dm. \]

It follows that \( h - g \) belongs to the closed linear manifold generated by the set \( \{ f - T^nf : f \in L^1(m) \} \). Thus
\[
\lim_{n} \left\| \frac{1}{n} \sum_{j=0}^{n-1} T^j h - g \right\|_1 = 0,
\]
and hence \( g \geq 0 \). Let \( A = \{ x \in X : g(x) = 0 \} \), and let the conservative and dissipative parts \([3]\) of \( T \) be \( C \) and \( D \), respectively. We shall first prove that \( A = D \). It is clear that \( D \subseteq A \). To see that \( A \subseteq D \), let \( T^* \) denote the corresponding adjoint operator on \( L^\infty(m) \). Since \( Tg = g \), it follows that \( T^*1_A \leq 1_A \), whence if we define \( B = A \cap C \) then \( T^*1_B = 1_B \) almost everywhere on \( C \) for each \( j \geq 0 \). Thus
\[
\int h1_B \, dm \leq \lim_{n} \int \left( \frac{1}{n} \sum_{j=0}^{n-1} h T^j 1_B \right) dm
\]
\[ = \lim_{n} \int \left( \frac{1}{n} \sum_{j=0}^{n-1} T^j h \right) 1_B \, dm = \int g1_B \, dm = 0. \]

Since \( h \) is strictly positive, it follows that \( m(B) = 0 \), and hence \( A \subseteq D \).

Let \( f \) be any function in \( L^1(m) \). Since \( A = D \), it follows at once that the limit \( (4) \) exists and is finite almost everywhere on \( A \). On the other hand, the Chacon-Ornstein theorem \([4]\) implies that
\[
\lim_{n} \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) = g(x) \lim_{n} \left( \frac{1}{n} \sum_{j=0}^{n-1} T^j f(x) / \sum_{j=0}^{n-1} T^j g(x) \right)
\]
exists and is finite almost everywhere on \( X \setminus A \). This completes the proof of the theorem.
It should be pointed out here that if $a_{n,j}$ is a regular matrix such that

$$\lim_{k \to \infty} \sum_{j=k}^{\infty} |a_{n,j+1} - a_{n,j}| = 0$$

uniformly in $n$ then it satisfies (1), (2) and (3) (see [5]).

**Remark 1.** Let $\{w_n; n \geq 1\}$ be a sequence of nonnegative numbers whose sum is one, and let $\{u_n; n \geq 0\}$ be the sequence defined by $u_n = w_1u_{n-1} + \cdots + w_nu_0$, $u_0 = 1$. Then the above argument together with Baxter’s ergodic theorem [2] implies that under the same condition as in the theorem, for any $f \in L^1(m)$ the limit

$$\lim_{n} \left( \sum_{j=0}^{n-1} u_j T^j f(x) \right) / \sum_{j=0}^{n-1} u_j$$

exists and is finite almost everywhere. The theorem is a special case of this result.

**Remark 2.** If $T$ maps, in addition, $L^p(m)$ into $L^p(m)$ and $\|T\|_p \leq 1$ for some $p$ with $p > 1$, then for any $f \in L^1(m)$ the limit (5) exists and is finite almost everywhere. This follows from [1] and [7].

**Bibliography**


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