CONCERNING COMPLETABLE MOORE SPACES

G. M. REED

Abstract. The author obtains a generalization of well-known theorems due to Younglove and Fitzpatrick concerning the existence of dense metrizable subspaces in complete and completable Moore spaces. Based on this result, a new class of noncompletable Moore spaces is presented. In particular, an example of a separable noncompletable Moore space is given.

A (complete) Moore space is a space which satisfies Axiom 0 and has a (complete) development satisfying the first three parts (all) of Axiom 1 in [5]. The completely regular (complete) Moore spaces are precisely the (Čech complete) semistratifiable $p$-spaces [3]. A Moore space is completable provided that some complete Moore space contains it as a subspace. Each complete Moore space which is metrizable is completely metrizable [9]. However, there are examples of Moore spaces which are not completable ([10], [11], [7] and [6]).

A development $G=(G_1, G_2, \ldots)$ for the Moore space $S$ is said to satisfy Axiom C at the point $p$ of $S$ if and only if, for each open set $D$ containing $p$, there is a positive integer $n$ such that each element of $G_n$ intersecting an element of $G_n$ which contains $p$ is contained in $D$. If $G$ is a development for the Moore space $S$, then $C(G)$, the set of all points at which $G$ satisfies Axiom C, is, if nonempty, a metrizable $G$-subset of $S$ ([4] and [12]).

In [12], Younglove proved that if $G$ is a development for the complete Moore space $S$ then $C(G)$ is dense in $S$. Fitzpatrick showed in [2] that a development $G$ for the completable Moore space $S$ need not satisfy Axiom C at any point of $S$ but that each completable Moore space does have a dense metrizable subspace.

The main result of this paper is Theorem 1: Each completable Moore space $S$ has a development $G$ such that $C(G)$ is dense in $S$. This improves Fitzpatrick's result in view of an example, given by the author in [8],

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of a Moore space $S$ which has a dense metrizable subspace but for which there exists no development $G$ such that $C(G)$ is dense in $S$. Using Theorem 1, the author is able to improve upon some other results concerning completable Moore spaces: (1) Theorem 2 establishes that each completable Moore space in which there does not exist an uncountable discrete collection of mutually exclusive open sets is separable. Armentrout had shown in [1] that each completable Moore space in which there does not exist an uncountable collection of mutually exclusive open sets is separable. (2) Theorem 3 establishes the existence of a separable non-completable Moore space. Ott in [6] had obtained the same result under the assumption of the continuum hypothesis which is not required here.

**Notation.** If $H$ is a collection of point sets, then $H^\omega$ denotes the union of the elements of $H$.

**Theorem 1.** Each completable Moore space $S$ has a development which satisfies Axiom C at each point of a dense subset of $S$.

**Proof.** Suppose $S$ is a subspace of the complete Moore space $Y$. Then $S$, regarded as space, is a complete Moore space. Thus, consider the complete development $G_1, G_2, \ldots$ for $S$.

For each positive integer $i$, let $H_i$ denote a maximal collection of mutually exclusive elements of $G_i$ such that $H_i^*$ is dense in $S$. For each element $R$ in $H_i$ and each positive integer $j$, let $R_j = \{p \in R | g \in G_j$ and $p \in g\}$, then $g$ is contained in $R$. Note that each $R_j$ is closed in $S$ and $R = \bigcup_{j=1}^\infty R_j$. By Theorem 162 in [5], no open set in a complete Moore space is the union of countably many closed sets no one of which contains a nonempty open set. Thus for each $R$ in $H_i$, denote by $u_R$ a nonempty open set in $S$ which is contained in $R_m$ for some positive integer $m$.

Now, if each of $i$ and $j$ is a positive integer, let $U_{ij} = \{u_R | R \in H_i$ and $u_R$ is contained in $R_j\}$. It follows that each $U_{ij}$ is a discrete collection of open sets in $S$. Thus, for each pair of positive integers $i$ and $j$, denote by $K_{ij}$ a subset of $S$ which contains exactly one point of $u_R \cap S$ for each $u_R$ in $U_{ij}$. Consider $K = \bigcup_{i=1}^\infty \bigcup_{j=1}^\infty K_{ij}$. Since $K$ is the union of countably many point sets $K_{ij}$ such that each is covered by a discrete collection of open sets intersecting it at only one point, it follows from the proof of Theorem 5 in [2] that there exists a development $G'$ for $S$ which satisfies Axiom C at each point of $K$. It remains only to show that $K$ is dense in $S$.

To see that $K$ is dense in $S$, consider the countable collection \{${H_i^*, H_j^*}$, \ldots\} of open sets each dense in the complete Moore space $S$. By the proof of Theorem 164 in [5], it follows that $\bigcap_{i=1}^\infty H_i^* = M$ where $M$ is a dense subset of $S$. But $M$ is contained in $K$. For if $p \in M$ and $p$ is contained in the open set $D$, there exists a positive integer $n$ such that each element of $G_n$ containing $p$ is contained in $D$. Since $p \in H_n^*$, then
$p \in R$ for some $R$ in $H_n$ and $u_R$, which contains a point of $K$, is contained in $D$. Thus $K$ is a subset of $S$ which is dense in $S$ and there exists a development $G'$ for $S$ which satisfies Axiom C at each point of $K$. It follows that $G_1'$, $G_2'$, $\cdots$, where for each $i$, $G_i' = \{ g \cap S | g \text{ in } G_i \}$ is a development for $S$ which satisfies Axiom C at a dense subset of $S$.

In [8], under the assumption of the continuum hypothesis, the author gave an example of a Moore space in which there exists an uncountable collection of mutually exclusive open sets but in which there exists no such collection that is also discrete. Thus Theorem 2 is a generalization of Theorem 2.1 in [1].

Theorem 2. Each completable Moore space $S$ in which there does not exist an uncountable discrete collection of mutually exclusive open sets is separable.

Proof. It follows from Theorem 1 that $S$ has a development $G$ such that $C(G)$ is dense in $S$. But in such a space, the existence of an uncountable collection of mutually exclusive open sets implies the existence of such a collection that is also discrete. For suppose that there exists an uncountable collection $H$ of mutually exclusive open sets in $S$. Let $M$ be a subset of $S$ containing one point of $h \cap C(G)$ for each $h$ in $H$. For each positive integer $i$, let $M_i = \{ p \in M | \text{ each element of } G_i \text{ intersecting an element of } G_i \text{ which contains } p \text{ is contained in the element of } H \text{ which contains } p \}$ and let $U_i = \{ \text{st}(p, G_i) | p \in M_i \}$. Note that for each $i$, $U_i$ is a discrete collection of mutually exclusive open sets. And since $M = \bigcup_{i=1}^\infty M_i$, there must exist a positive integer $k$ such that $U_k$ is uncountable.

Thus the completable Moore space $S$ satisfies the hypothesis of Theorem 2.1 in [1] and is therefore separable.

Theorem 3. There exists a separable, noncompletable Moore space $X$.

Proof. In [8] the author gave an example of a Moore space $S$ for which there exists no development $G$ such that $C(G)$ is dense in $S$. Thus, by Theorem 1, it suffices to show that there exists a separable Moore space $X$ which has $S$ as a subspace. Such a space $X$ will now be constructed by "sewing" onto $S$ countably many tangent disc spaces (i.e., Neimyitzki planes).

1. Points of $S$. The points of $S$ are precisely all sequences $(p_1, p_2, \cdots, p_k, \cdots)$ of nonnegative real numbers such that $p_k > 0$, $p_i$ is rational for $i < k$, and $p_i = 0$ for $i > k$. For convenience we will express a point $p$ of $S$ as $p = (p_1, p_2, \cdots, p_k, 0, \cdots)$ where $k$ is the greatest integer such that $p_k$ is positive.
2. Regions (basic open sets) of S. Suppose \( n \) is a positive integer and \( p=(p_1, p_2, \cdots, p_k, 0, \cdots) \) is a point of \( S \). (i) If \( p_k \) is irrational, then let

\[
\begin{align*}
r_{n}(p) &= \{(p_1, p_2, \cdots, p_{k-1}, t^1_{k}, t^1_{k+1}, 0, \cdots) \text{ in } S \mid \ 0 \leq t^1_{k+1} \leq 1/n, t^1_{k} = p_k + t^1_{k+1} \text{ and } t^1_{k} \text{ is rational}\}, \\
r_{n}(p) &= \{(p_1, p_2, \cdots, p_{k-1}, t^2_{k}, t^2_{k+1}, t^2_{k+2}, 0, \cdots) \text{ in } S \mid \text{ there exists } (p_1, p_2, \cdots, p_{k-1}, t^1_{k}, t^1_{k+1}, 0, \cdots) \text{ in } r_{n}(p), \ 0 \leq t^2_{k+2} \leq 1/n, \\
&\quad t^2_{k+1} = t^1_{k+1} + t^2_{k+2}, \text{ and } t^2_{k+1} \text{ is rational}\},
\end{align*}
\]

and continue this process to define \( r_{n}(p) \) for each positive integer \( j \). Let \( g_n(p) = \bigcup_{j=1}^{n} r_{n}(p) \cup \{p\} \). (ii) If \( p_k \) is rational, let

\[
\begin{align*}
r_{n}(p) &= \{(p_1, p_2, \cdots, p_{k-1}, t^1_{k}, t^1_{k+1}, \cdots, t^1_{k+i-1}, t^1_{k+i}, 0, \cdots) \text{ in } S \mid \text{ there exists } (p_1, p_2, \cdots, p_{k-1}, t^1_{k}, t^1_{k+1}, \cdots, t^1_{k+i-1}, t^1_{k+i}, 0, \cdots) \text{ in } r_{n}(p), \ 0 \leq t^1_{k+i} \leq 1/n, \\
&\quad t^1_{k+i-1} = t^1_{k+i} + t^1_{k+i}, \text{ and } t^1_{k+i-1} \text{ is rational}\},
\end{align*}
\]

and continue this process to define \( r_{n}(p) \) for each positive integer \( j \). Let \( g_n(p) = \bigcup_{j=1}^{n} r_{n}(p) \cup \{p\} \). Now, \( g \) is a region for \( S \) if and only if there exist a positive integer \( n \) and a point \( p \) of \( S \) such that \( g = g_n(p) \).

For each positive integer \( i \), let \( G_i = \{g_i(p) | p \in S\} \). For each positive integer \( j \), let \( G_j = \bigcup_{i=1}^{j} G_i \). It follows that \( S \) is a Moore space and \( G_1, G_2, \cdots \) is a development for \( S \).

3. Construction of \( X \). As noted in [8], \( S \) is the sum of countably many “lines” where each line can be expressed as \( \{(p_1, p_2, \cdots, p_k, y, 0, \cdots) \text{ in } S \mid y \text{ is a positive real number}\} \) where \( k \) is a positive integer and \( p_i \) is a fixed rational number for \( 1 \leq i < k \) if \( k > 1 \).

Consider the subset \( M \) of the upper plane such that \( M = \{(x, y) | x > 0 \text{ and } y \geq 0\} \). To each line \( L \) as above associate a unique copy \( M_L \) of \( M \) such that \( L \) is identified with the nonnegative x-axis in \( M_L \).

Now for each line \( L \), consider \( M_L \) as a subset of the plane with the usual topology. Suppose \( n \) is a positive integer and \( p \) is a point of \( M_L \). (i) If \( p \) is a point in \( M_L \) not in \( L \), let \( g_n(p) \) denote the common part of \( M_L \) and
the interior of a circle about \( p \) with radius equal to the lesser of \( 1/n \) and the ordinate of \( p \). (ii) If \( p=(p_1, p_2, \ldots, p_k, 0, \ldots) \) is a point of \( L \) such that \( y \) is irrational, let \( u_n(p) \) denote the common part of \( M_L \) and the interior of a circle of radius \( 1/n \) lying wholly above the \( x \)-axis and tangent to the \( x \)-axis at \( p \) together with the point \( p \).

Note that for each point \( p=(p_1, p_2, \ldots, p_k, 0, \ldots) \) in \( S \) such that \( p_k \) is irrational, \( u_n(p) \) is uniquely defined for each positive integer \( n \). Also, since there are only countably many lines in \( S \), \( \{M_L| L \text{ a line in } S \} \) is countable.

4. Points of \( X \). Let \( X \) be the set to which \( p \) belongs if and only if \( p \) is a point of \( S \) or \( p \) is a point of \( M_L \) for some line \( L \) in \( S \).

5. Regions of \( X \). Suppose \( n \) is a positive integer and \( p \) is a point of \( X \).
(i) If \( p=(p_1, p_2, \ldots, p_k, 0, \ldots) \) is a point of \( S \) such that \( p_k \) is irrational, let

\[
h_n(p) = g_n(p) \cup \left( \bigcup \{u_n(q) \mid q \in g_n(p)\} \right).
\]

(ii) If \( p=(p_1, p_2, \ldots, p_k, 0, \ldots) \) is a point of \( S \) such that \( p_k \) is rational, let

\[
h_n(p) = g_n(p) \cup \left( \bigcup \{h_n(q) \mid q \in g_n(p) \text{ and } q = (q_1, q_2, \ldots, q_{m}, 0, \ldots) \text{ in } S \text{ where } q_{m} \text{ is irrational}\} \right).
\]

(iii) If \( p \) is a point of \( M_L \), for some line \( L \) and \( p \) is not in \( L \), let \( h_n(p)=g_n(p) \).

6. Properties of \( X \). For each positive integer \( i \), let \( H_i = \{h_i(p) \mid p \in S\} \). For each positive integer \( j \), let \( H_j = \bigcup_{i=1}^{\infty} H_i \). It follows that \( X \) is a Moore space and \( H_1, H_2, \cdots \) is a development for \( X \). Also, since each open set in \( X \) contains a subset which is open (with respect to the topology of the plane) in \( M_L \) for some line \( L \) in \( S \), then \( X \) is separable. Finally, the space \( S \) is a subspace of \( X \), thus \( X \) is not completable.

**References**


Department of Mathematics, Auburn University, Auburn, Alabama 36830

Department of Mathematics, Ohio University, Athens, Ohio 45701 (Current address)