KRONECKER SETS AND METRIC PROPERTIES
OF $M_0$-SETS
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Abstract. A method for constructing both sets of multiplicity
and Kronecker sets within a given set of multiplicity is derived from
the work of Ivashev-Musatov; it is shown that the Hausdorff
measures and other measures are essentially distinct. Finally, an
improvement of a theorem of Salem is obtained, using Pyateckii-
Shapiro's theorem on non-$M$ sets.

1. The class of complex Borel measures $\mu$ on the real axis, such that
$\hat{\mu}(u)\to 0$ as $|u|\to \infty$, is denoted $R$, and a closed set is called $M_0$ if it supports
some measure $\mu \neq 0$ of class $R$. With a measure $\mu$ the entire space $L^1(\mu)$ is
contained in $R$ [7, I, p. 143], so that we mostly study probability measures
in $R$. The most striking examples of non-$M_0$ sets are the Kronecker sets:
A compact linear set $E$ is a $K$-set if each continuous function on $E$ of
modulus 1 admits uniform approximation on $E$ by characters $\chi(x) =
eel u \in (-\infty, u < \infty)$ ([2], [6]).

An easy application of Fubini's theorem shows that if $E_1$ and $E_2$ are
closed, $E_1$ is $M_0$ and $m(E_2) > 0$, then $E_1 \cap (E_2 + x)$ is $M_0$ for an $x$-set of
positive Lebesgue measure. A sharp converse is true.

Theorem 1. Let $F$ be a closed set of Lebesgue measure $m(F) = 0$. Then
there exists an $M_0$-set $E$ so that $(F + x) \cap E$ is a $K$-set for every real $x$.

Theorem 1 is a consequence of a more general assertion; for closed
sets $F$ and numbers $r > 0$ let $m(F, r) = m\{x: \text{dist}(x, F) \leq r\}$.

Theorem 1'. Let $g$ be any positive, increasing function on $(0, 1)$ and
let $g(0+) = 0$. Then there is an $M_0$-set $E$ such that each closed set $F \subseteq E$
is a $K$-set, unless $m(F, r) > g(r)$ for all $r \leq r_0(F)$.

Theorem 1' is derived from a very general theorem in which Lebesgue
measure plays no special role.

Theorem 2. Let $\mu$ be a probability of class $R$ and of closed support $S$;
let $(T_k)_{k=1}^\infty$ be a sequence of closed sets with $\lim \mu(T_k) = 0$. Then there exists an
$M_0$-set $S_0 \subseteq S$ with this property: for every infinite set $I$ of integers $k$, $S_0 \cap \bigcap_{k} T_k$ is a $K$-set.

The proof of Theorem 2 requires a few preliminary remarks and adjustments.

(a) There are open sets $W_k \supseteq T_k$ whose boundaries $\partial W_k$ have $\mu$-measure 0, and moreover $\mu(W_k) < \mu(T_k) + k^{-1}$. This much is true for Baire measures in any metric space. In the set of probability measures in $L_1(\mu)$, the mappings $\lambda \to \lambda(W_k)$ are $w^*$-continuous, because $\mu(\partial W_k) = 0$. In the same direction, let $V$ be an open set; the set of probabilities such that $\lambda(V) = 0$ is $w^*$-closed.

(b) A distance function on the Borel measures is defined by the formula $p(\lambda_1, \lambda_2) = \sup |\lambda_1(W_k) - \lambda_2(W_k)|$. In the proof we shall construct probability measures $\mu_j$ in $L^1(\mu)$ such that $\mu = \mu_0$ and

(i) $p(\mu_j, \mu_{j+1}) \leq 2^{-i}$ for $j = 0, 1, 2, \cdots$.

(ii) $|\hat{\mu}_j(u) - \hat{\mu}_{j+1}(u)| \leq 2^{-i}$ for every $u$, with at most one exception $j = j(u)$.

(iii) $\|\mu_{j+1} - \mu_j\| \leq 2\mu_j(W_j)$.

From (i) it will follow that $\lim \mu_j(W_j) = 0$ and then from (ii) and (iii) that $\hat{\mu}_j$ converges uniformly on the real axis, to a function also tending to 0 at infinity. The property $\mu(\partial W_j) = 0$ is used for (i).

(c) Let $\lambda$ be a probability measure of class $\mathcal{R}$, and $f$ a real measurable function. Then the measurable functions of $x$, $ux - f(x)$, tend to uniform distribution (modulo 1) in $\lambda$-measure, as $|u| \to \infty$. To prove this we use Weyl's criterion [1, IV] for uniform distribution, with the notation $e(t) \equiv \exp 2\pi i t$. It must be proved that

$$\lim \int e(kux - kf(x)) \lambda(dx) = 0 \quad \text{for } k = 0, 1, 2, \cdots.$$ 

But the complex measures $e(-kf(x)) \lambda(dx)$ are of class $\mathcal{R}$ and this fact expresses the limit relation required.

(d) Let now $S$ be compact and totally disconnected, as is easily achieved, and $(g_k)_1^\infty$ be a dense sequence in the real Banach space $C(S)$. We shall use the fact that, on $S$ and any closed subset of $S$, a continuous function of modulus 1 admits uniform approximation by the subset $e(g_k)$. We construct closed sets $A_j \subseteq W_j$ and measures $\mu_j$ by the formulas

$$\mu_{j+1}(B \sim W_j) \equiv c_j \mu_j(B \sim W_j), \quad \mu_{j+1}(B \cap W_j) \equiv c_j \mu_j(B \cap A_j)$$

for appropriate constants $c_j > 0$, so that $\mu_{j+1}$ is again a probability. As $\mu_{j+1}$ and $\mu_j$ differ only by a certain adjustment outside of $W_j$, an easy calculation yields (iii). Moreover, to each function $g_1, \cdots, g_j$ there will be integers $u_1, \cdots, u_j$ so that $|\mu_k(x) - g_k(x)| < 2^{-j}$ (modulo 1), $|u_k(x) - g_k(x)| < 2^{-j}$ (modulo 1), on $A_j$. Observe also that the measures $\mu_j$ have decreasing
closed supports; this, with the last remark in (a), shows that any w*-accumulation point \( \lambda \) of the sequence \( (\mu_j) \) has support \( S_0 \) such that \( S_0 \cap T_j \leq S_0 \cap \wedge_j A_j \) so that \( S_0 \) has the 'Kronecker' properties asserted.

The measures \( \mu_j \) are constructed by induction; suppose this accomplished up to \( \mu_s \). Because \( \mu_0, \ldots, \mu_s \) are of class \( R \) (belonging to \( L^1(\mu) \)), there is a number \( Y \) such that \( |\mu_s(u)| < 2^{-s-1} \) for \( 0 \leq j \leq s \) whenever \( |u| \geq Y \). Thus requirement (ii) need only be attained on the interval \( |u| \leq Y \).

(e) Let \( 0 < \delta < 2^{-s-1} \), and let \( (W_n) \) be a finite partition of \( W_s \) into subsets of diameter \( < \delta \). When the measure \( \mu_{s+1} \) has the property that \( \mu_{s+1}(W_n) = \mu_s(W_n) \) for each \( n \), then \( |\mu_{s+1}(u) - \mu_s(u)| \leq \delta |u| < 2^{-s} \) on \( [-Y, Y] \). It remains to construct the set \( A_s \) and attain inequality (i).

Using Weyl's criterion \( s \) times in succession, we find that there are integers \( u_1, \ldots, u_s \) so that the set \( \{|u_j - g_j(x)| < 2^{-s} \pmod{1}, 1 \leq j \leq s \} \) meets each \( W_n \) in a set of \( \mu_s \)-measure at least \( 2^{-s-1} \mu_s(W_n) = a_s \mu_s(W_n) \). Here \( a_s \) depends only on \( s \). Then, using the regularity of Borel measures, there is a closed set \( A_s \) on which the required approximations are valid, while \( \mu_s(W_n \cap A_s) = a_s \mu_s(W_n) \). Thus \( \mu_{s+1} \) can be defined with \( c_s = 2a_s^{-1} \) (or, in fact, any number \( c_s > 2^s \)).

(f) To obtain inequality (i) we note that all the measures \( \mu_{s+1} \) arising in this construction satisfy the same inequality \( 0 \leq \mu_{s+1} \leq c_s \mu_s \) (as set functions), whence \( \mu_{s+1}(W_k) \leq c_s \mu_s(W_k) \) for all measures \( \mu_{s+1} \). Thus the inequality \( |\mu_{s+1}(W_k) - \mu_s(W_k)| \leq 2^{-s} \) is a constraint only when \( c_s \mu_s(W_k) \geq 2^{-s-1} \), that is for \( k \leq k_s \), say. For these values of \( k \), we use the w*-continuity of the mappings \( \lambda \mapsto \lambda(W_k) \) on \( L^1(\mu) \), by making \( \delta > 0 \) small enough to bring each \( \mu_{s+1} \) close to \( \mu_s \) in the w*-topology. This completes the construction of \( \mu_{s+1} \) and the proof of Theorem 2. It is worthwhile remarking that if any sequence \( e(u_m(x)) \) of characters is assigned in advance, with \( |u_m| \to \infty \), then the Kronecker property of the intersections \( S_0 \cap \cap_k T_k \) can be enforced using only these characters.

**Proof of Theorem 1'.** Let \( S \) be an interval of length 1 and \( \mu \) the Lebesgue measure on \( S \). Let \( J(n) \) be the usual division of \( S \) into intervals of length \( n^{-1} \), \( C(n) \) the set of unions of at most \( ng(2n^{-1}) \) of these intervals, and finally let \( (T_k) \) be an enumeration of \( C(n) \). To see that Theorem 2 applied with the present choice of \( \mu \) and \( (T_k) \) yields Theorem 1', consider a subset \( F \subseteq S \) such that \( m(F, r) \leq g(r) \) for some \( r \) in \( (0, 1) \). Defining \( n \) by the inequalities \( n^{-1} < r \leq (n - 1)^{-1} \), we see that \( F \) meets at most \( nm(F, r) \leq ng(r) \leq ng(2n^{-1}) \) intervals from \( J(n) \), and Theorem 1' is proved.

To see the special properties of the measure function \( m(F, r) \), we compare it with Hausdorff measure \([5, 11]\). Recall that if \( h \) is increasing and positive on \((0, \infty)\) and \( h(0+) = 0 \), a set \( F \) has \( h \)-measure \( 0 \) if, to each \( \varepsilon > 0 \), there is a division \( F = UF_i \), so that \( \sum h(\text{diam } F_i) < \varepsilon \). By almost the same arguments as before, the following improvement of the theorem of Ivashev-Musatov is obtained \([3]\).
Theorem 3. Each $M_0$-set $E$ contains an $M_0$-set $E_1$ of $h$-measure 0.

2. Theorems 1' and 3 are in obvious contrast, and another contrast to Theorem 3 is obtained from Dirichlet's theorem on simultaneous approximation ([1, I]; [7, II, p. 153]).

Theorem 4. Let $\mu$ be a probability of class $R$ and $F_k$ any union of $C \log k$ intervals of length $k^{-1}$. Then $\lim \mu(F_k) = 0$.

Proof. We shall first show that $\lim \sup \mu(F_k) \leq 2 \exp -C^{-1}$. In fact let $\exp -C^{-1} < \delta < 1$ and $\delta > 0$, and let $x_1, \cdots, x_N (N \leq C \log k)$ be the centers of the given intervals. The inequalities

$$|\xi x_j| \leq b \pmod 1, \quad 1 \leq j \leq N, |\xi| \leq \delta k,$$

have an integer solution $\xi > 0$ as soon as $N \log b + \log \delta + \log k > 0$, and in fact the largest solution, say $\xi_k$, tends to $+\infty$ because $C \log b > -1$. (See the remarks below on Dirichlet's theorem.) Because $1 \leq \xi_k \leq \delta k$ we find $|\xi_k x| \leq b + \delta$ (modulo 1) for every $x$ in $F_k$ and by Weyl's criterion, $\lim \sup \mu(F_k) \leq 2(b + \delta)$.

Next, for any integer $r \geq 1$, the intervals comprised in $F_k$ can be divided into $r$ approximately equal classes; this yields the estimate

$$\lim \sup \mu(F_k) \leq 2r \exp -rC^{-1},$$

whence $\lim \mu(F_k) = 0$.

This theorem is to be compared with Theorem 3, for it gives a sort of modulus of continuity. In particular, $F$ is non-$M_0$ provided $\lim \inf m(F, r)/r \log r^{-1} < \infty$. This result is very sharp, as shown by a theorem of Kahane [4]. (Theorem 4 resolves the question raised at the conclusion of [4].)

In both Theorem 4 and the following argument, it is necessary to obtain a large number of solutions in Dirichlet's theorem on simultaneous approximation. The references cited deduce Dirichlet's theorem from Minkowski's theorem on lattice points in a symmetric convex set $S$ in the Euclidean space $R^n$. However, if $2^{-n}V(S)$ is large, the analysis of Minkowski's theorem yields a large number of lattice points, and these determine distinct solutions in the Dirichlet theorem.

Besides the concept of $M_0$-set, there is also that of $M$-set, defined as follows (for closed sets, at any rate). A closed set $E$ is an $M$-set if it supports a distribution $T \neq 0$ whose Fourier transform $\hat{T}$ belongs to $L^\infty$ and vanishes at infinity. This is plainly not the language of Riemann and Cantor; a more traditional definition is adopted in [7], and the equivalence of the two theories is explained in [5, V]. The following theorem describes a stronger property of a closed set than that deduced from Theorem 4,
The proof requires knowledge of the work of Pyateckiï-Shapiro [7, I, p. 346], especially the definition of an $H^{(r)}$-set. To obtain this theorem, we observe that $F$ is contained in sets of type $F_k$, for a sequence of integers $k$ tending to infinity.

Given the constant $C>0$, let $r>C \log 2$ be an integer, and $0<b<\frac{1}{2}$ while $\log b>-rC^{-1}$. The integers $j$ in the interval $[1, C \log k]$ are divided into $r$ nearly equal classes, say $L_1, \ldots , L_r$. The Diophantine inequalities, for $n=1, \ldots , r$,

$$\left| x_j \right| \leq \frac{1}{2}(1-2b)k \equiv \delta k,$$

have solutions $\xi_n$ forming a set $Q_n$, whose cardinalities tend to infinity. Here of course the inequality $-\log b<rC^{-1}$ is decisive. Compare [5, p. 95].

To apply Pyateckiï-Shapiro's method [7, I, p. 346] we must choose special $r$-tuples $(\xi_1, \ldots , \xi_r)$ from $Q=Q_1 \times \cdots \times Q_r$. More exactly, given any finite union $U$ of linear inequalities of the form

$$a_0 + a_1 x_1 + \cdots + a_r x_r = A, \quad \max |a_i| \leq 1,$$

we must choose a point $(\xi_1, \ldots , \xi_r)$ in $Q \sim U$. Now in one of our inequalities, with (say) $|a_r| \geq 1$, any $\xi_1, \ldots , \xi_{r-1}$ admits at most $2A+1$ choices for $\xi_r$, so $U$ meets $Q$ in a very small part, that is $|Q \cap U| = o(|Q|)$ so that $Q \sim U$ is nonvoid for large $k$.

Consequently, a set $F$ is of type $H^{(r)}$ provided it is contained in $C \log k$ intervals of length $k^{-1}$, for an unbounded sequence of integers $k$. This conclusion, with $C \log k$ strengthened to $o(\log k)$, is due to Salem [5, p. 95]; [4] and [5] treat slightly different properties, and for these $o(\log k)$ is the critical case.

REFERENCES


