SELECTION OF REPRESENTING MEASURES
FOR INNER PARTS

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Abstract. If a compact convex set $K$ has an inner part $\Delta$ then there is a selection of pairwise boundedly absolutely continuous representing measures for $\Delta$ if and only if $K$ is finite dimensional.

Let $K$ denote a compact convex set in a LCTVS, $A(K)$ the affine continuous real functions on $K$, $\mathcal{P}(K)$ the set of regular Borel probability measures on $K$. Let $\Phi: \mathcal{P}(K) \to K$ be the map which associates to each measure $\mu$ its barycentre. Then $\Phi$ is affine, weak* continuous, and onto $K$. If $\Phi(\mu) = x$ we say $\mu$ represents $x$.

If $L$ is any convex set, $x, y \in L$ and $r > 0$, we say $[x, y]$ extends by $r$ in $L$ if $x + r(x - y) \in L$ and $y + r(y - x) \in L$. We write $x \sim y$ if $\exists r > 0$ such that $[x, y]$ extends by $r$ in $L$. This is an equivalence relation on $L$ and the equivalence classes are the parts of $L$. It is easy to show that $\Phi$ carries parts into parts: If $\Pi$ is a part of $\mathcal{P}(K)$ then $\Phi(\Pi)$ is contained in a part of $K$. Conversely if $\Delta$ is a part of $K$ and $F$ is any finite subset of $\Delta$ then there exists a part $\Pi$ of $\mathcal{P}(K)$ such that $F \subset \Phi(\Pi)$. Indeed if $F = \{x_1, x_2, \ldots, x_n\}$ choose $y_i$ and $z_i$ in $K$ such that $x_i \in (y_i, z_i)$, the open line segment with endpoints $y_i$ and $z_i$, and $x_i \in (y_i, z_i)$ ($2 \leq i \leq n$). If $\Phi(\mu_i) = y_i$ and $\Phi(\nu_i) = z_i$ for $\mu_i, \nu_i \in \mathcal{P}(K)$, then the part $\Pi$ containing $\sum (\mu_i + \nu_i)/(2n - 2)$ satisfies $F \subset \Phi(\Pi)$. Indeed since $x_i \in (y_i, z_i)$ for each $i$, we can clearly find a representing measure $\omega$ for $x_i$ in $\Pi$. Since $x_i \in (y_i, x_i)$, an affine combination of $\mu_i$ and $\omega$ yields a representing measure for $x_i$ in $\Pi$.

Thus if $\Delta$ is a part of $K$ one might ask whether

$$\Delta = \Phi(\Pi) \quad \text{for some part } \Pi \text{ of } \mathcal{P}(K).$$

Indeed Bear posed this question in [3] and reproduced an example of Har’kova [4] to show that (1) need not hold if $\mathcal{P}(K)$ is replaced by $\mathcal{P}(\Gamma)$ where $\Gamma$ is the Shilov boundary of $A(K)$.

Since two probability measures $\mu$ and $\nu$ on $K$ are in the same part of $\mathcal{P}(K)$ if and only if $\mu \leq k\nu$ and $\nu \leq k\mu$ for some $k$, condition (1) asserts...
the existence for $\Delta$ of a selection of representing measures on $K$ which are pairwise boundedly absolutely continuous. There are two special cases when (1) is true for all parts $\Delta$ of $K$. One is when $K$ is a simplex, for then there are unique maximal representing measures [6, §9], the other when $K$ is finite dimensional (Theorem 1).

Let $K' = \{x \in K : (\forall y \in K)(\exists r > 0)x + r(x - y) \in K\}$. It can happen that $K' = \emptyset$, but if $K' \neq \emptyset$ it is a part of $K$ called the inner part. Finite dimensional convex sets, for example, always have nonempty inner parts. In Theorem 1 we show that if $\Delta = K' \neq \emptyset$ then (1) holds for $\Delta$ if and only if $K$ is finite dimensional.

First some preliminaries. If $L$ is a compact convex set, $x, y \in L$ and $x \sim y$; let

$$d(x, y) = \inf\{\log(1 + 1/r) : [x, y] \text{ extends by } r\}.$$ 

In [3, Lemma 3.4] it is shown that $d$ is a metric on each part of $L$, called the part metric. Now denote by $d$ and $D$ the part metrics on $K$ and $\mathcal{P}(K)$ respectively and let

$$b(x, r) = \{y \in K : d(x, y) \leq r\} \quad \text{and} \quad B(\mu, r) = \{v \in \mathcal{P}(K) : D(\mu, v) \leq r\}.$$ 

**Lemma 1.** Suppose $\Delta$ is a part of $K$, $\Pi$ a part of $\mathcal{P}(K)$, and $\Delta = \Phi(\Pi)$. Then there exist $\mu \in \Pi$ and positive numbers $M$ and $k$ such that if $x = \Phi(\mu)$ then

$$b(x, \log(1 + 1/M)) \subseteq \Phi(B(\mu, \log k)).$$

**Proof.** If $v \in \Pi$ then the sets $\Phi(B(v, r))$ are closed in the part metric topology. Indeed suppose $x_n = \Phi(\mu_n)$ with $\mu_n \in B(v, r)$ and $d(x_n, x) \to 0$. Choose a subset $\mu_{n_k}$ converging weak* to $\mu$. Since $B(v, r)$ is weak* closed (easy to check), $\mu \in B(v, r)$. Since $\Phi$ is weak* continuous, $x_{n_k}$ converges in $K$ to $\Phi(\mu)$. But since $d(x_{n_k}, x) \to 0$, $x_{n_k}$ converges in $K$ to $x$, hence $x = \Phi(\mu) \in \Phi(B(v, r))$. (It is an easily verified general fact that in any part of a compact convex set the part metric topology is stronger than the relativized compact topology.)

Since $\Delta = \Phi(\Pi) = \bigcup_{n=1}^{\infty} \Phi(B(v, n))$ and the part metric on $\Delta$ is complete [1, §3], the Baire category theorem tells us that we can find $x \in \Delta$ and integers $h$ and $M$ such that $b(x, \log(1 + 1/M)) \subseteq \Phi(B(v, h))$. Choose $\mu \in \Pi$ such that $\Phi(\mu) = x$ and choose $k$ such that $B(v, h) \subseteq B(\mu, \log k)$. \qed

**Lemma 2.** Suppose $x \in K'$. Then $\exists \delta > 0$ such that

$$y \in K \Rightarrow x + \delta(x - y) \in K.$$ 

**Proof.** Let $H = K - x$. Then $0 \in H$ and so $H \cap -H$ is closed, convex and absorbs each point of $H$ and $-H$. Since $H$ is compact, convex,
$H \cap -H$ absorbs $H$ [5, Corollary 10.2]. Thus $\exists \delta > 0$ such that $\delta H \subset H \cap -H \subset -H$. Thus

$$y \in K \Rightarrow y - x \in H \Rightarrow \delta (y - x) \in -H \Rightarrow x + \delta (x - y) \in K.$$  \qed

If $A$ is a normed linear space and $\varepsilon \geq 0$ let $B_\varepsilon = \{ x \in A : ||x|| \leq \varepsilon \}$.

**Lemma 3.** Suppose $E$ is a normed linear space and $G$ is a weak* closed subspace of the dual space $E^*$. Suppose $x \in E^*$, $r \geq 0$ and $(x + B_r) \cap G = \emptyset$. Then $\exists f \in E$ such that $||f|| = 1$, $f(G) = 0$ and $f(x) > r$.

**Proof.** Since $x + B_r$ is weak* compact and $G$ is weak* closed. Hence $\exists f \in E$ such that $||f|| = 1$, $f(G) < \alpha$ and $f(x + B_r) \supseteq \alpha$ for some $\alpha$. Since $G$ is a subspace, $\alpha > 0$ and $f(G) = 0$. Since $||f|| = 1$ we can find $y \in B_r$ such that $f(y) > r - \alpha$. Then $x - y \in x + B_r$ so $f(x - y) \geq \alpha$ hence $f(x) \geq \alpha + f(y) > r$. \qed

Now for the main theorem. We always think of $K$ as embedded in the Banach space $A(K)^*$ with the weak* topology. The norm of $A(K)^*$ provides a metric topology on $K$ which we will refer to as the norm topology.

**Theorem 1.** Suppose $\Delta = K \neq \emptyset$. Then the following are equivalent.

1. $A = \Phi(\Pi)$ for some part $\Pi$ of $\mathcal{P}(K)$.
2. $K$ is finite dimensional.

**Proof.** (1) $\Rightarrow$ (2). Suppose (1) and suppose that $K$ is metrizable. We will show that, in this case, $K$ is finite dimensional. Then we will reduce the general case to this one.

We first show that $K$ is norm separable. If $\mu \in \Pi$ then $\Pi \subset L^1(\mu)$ (via Radon Nikodym), and the norm topology that $\Pi$ gets from $L^1(\mu)$ is the same as the norm topology it gets as a subset of $C(K)^*$. Indeed if $g, h \in L^1(\mu)$ then

$$\sup_{f \in \mathcal{C}(K) : ||f||_\infty = 1} \int f (g - h) \, d\mu = ||g - h||_1,$$

where $||\cdot||_1$ denotes the variation norm in the Banach space $\mathcal{M}(K)$ of Radon measures on $K$. Since $L^1(\mu)$ is separable ($K$ is metrizable), $\Pi$ is norm separable in $C(K)^*$. Since $\Phi$ is the restriction to $\mathcal{P}(K)$ of the natural, norm-decreasing surjection $\Phi : C(K)^* \rightarrow A(K)^*$, $\Delta = \Phi(\Pi)$ is norm separable. Since $\Delta = K^i$ is norm dense in $K$, $K$ is norm separable.

Now we show that $K$ is norm compact. Since $K$ is norm complete it will be enough to find for any $\varepsilon > 0$ a finite set $F \subset A(K)^*$ such that $K \subset F + B_{2\varepsilon}$. So suppose $\varepsilon > 0$. Choose $\mu$, $M$, $k$ and $x$ from Lemma 1 and $\delta$ from
Lemma 2 so that $\delta(1+1/M) \leq 1$. Since $K$ is norm separable, we can cover $K$ with countably many balls of norm radius $r = \varepsilon\delta/2M$. A finite number of these balls contains all but at most $\gamma = \varepsilon\delta/2kM$ of the measure $\mu$. Let $P$ be a finite dimensional subspace of $A(K)^*$ containing $x$ and the centres of these finitely many balls.

We claim that $K \subset P + B_\varepsilon$. Indeed suppose $y \in K$ but $y \notin P + B_\varepsilon$. Let $z = x + (\delta/M)(y - x)$. Then $z \in K$ and $d(x, z) \leq \log(1 + 1/M)$. Indeed $x + M(x - z) = x + \delta(x - y)$ which is in $K$ by Lemma 2, and $z + M(z - x) = x + \delta(1 + 1/M)(y - x)$ which is in $K$ since $\delta(1 + 1/M) \leq 1$. So by Lemma 1 we can choose $v \in B(\mu, \log k)$ such that $z = \Phi(v)$. An easy computation shows that $dv = g d\mu$ with $1/k \leq g \leq k$. Also, since $P$ is weak* closed and $y \notin P + B_\varepsilon$ we can find $f \in A(K)$ such that $\|f\| = 1$, $f(P) = 0$ and $f(y) > \varepsilon$ (Lemma 3). Then

$$f(z) = (\delta/M)f(y) > \varepsilon\delta/|M|,$$

and

$$v(f) = \int \int fg d\mu = \int_{|f| \geq r} fg d\mu + \int_{|f| > r} fg d\mu \leq r \int g d\mu + \|f\| \cdot \mu(|\{f| > r}) \leq r + k\gamma = \varepsilon\delta/|M|$$

(where $\mu(|\{f| > r}) \leq \gamma$ since $|f(w)| > r \Rightarrow w \notin P + B_\varepsilon$). Since $f \in A(K)$ and $\Phi(v) = z$ we must have $v(f) = f(z)$, a contradiction.

So $K \subset P + B_\varepsilon$. Hence $K = [(K + B_\varepsilon) \cap P] + B_\varepsilon$. Now $(K + B_\varepsilon) \cap P$ is finite dimensional and norm bounded, so relatively norm compact, and we can choose a finite set $F \subset A(K)^*$ so that $F + B_\varepsilon$ contains it. Hence $K \subset F + B_\varepsilon$.

So $K$ is norm compact. We deduce that the unit ball $B_1$ of $A(K)^*$ is norm compact. Indeed it follows from the Hahn Banach Theorem that every element of $A(K)^*$ is given by a Radon measure on $K$. Use the Hahn decomposition of this measure and the fact that any probability measure on $K$ has a barycentre in $K$ to deduce that, for any $\lambda \in B_1$, there exists $k, h \in K$ and $0 \leq x, \beta \leq 1$ such that

$$\lambda(f) = \alpha f(k) - \beta f(h) \quad (f \in A(K)).$$

Thus $B_1$ is contained in a continuous image of $K \times K \times [0, 1] \times [0, 1]$, and is norm compact. It follows that $A(K)^*$ is finite dimensional, and so is $K$.

Now drop the metrizability assumption; suppose $K$ has (1) but is not finite dimensional. Choose a countably infinite, linearly independent sequence $\{f_n\} \subset A(K)$ such that $\|f_n\| \leq 2^{-n}$. Define the map $\Psi$ from $K$ into $l^2$ by $\Psi(x) = f_n(x)$. $\Psi$ is affine and continuous, hence maps $K$ onto a compact convex subset $H$ of $l^2$. From Lemma 4 below $H^i = \Psi(K)^i \neq \emptyset$. Since every $x \in K^i$ has a representing measure in $\Pi$, every $h \in H^i$ has a...
representing measure in $\Pi \circ \Psi^{-1} = \{\mu \circ \Psi^{-1} : \mu \in \Pi\}$. Since $\Pi$ is a part of $\mathcal{P}(K)$, $\Pi \circ \Psi^{-1}$ is contained in a part of $\mathcal{P}(H)$ (from linearity of the map $\mu \mapsto \mu \circ \Psi^{-1}$). So $H$ has property (1) and since it is metrizable it is, by the first part of the proof, finite dimensional. This contradicts the linear independence of $\{f_n\}$.

**Lemma 4.** Suppose $K$ and $H$ are convex sets and $K' \neq \emptyset$. Suppose $\Psi : K \to H$ is affine and onto. Then $H' = \Psi(K')$.

**Proof.** Clearly $\Psi(K') \subseteq H'$. Assume $x \in H'$; Choose $z' \in K'$ and let $z = \Psi(z')$. Since $x \in H'$, $x = \lambda z + (1 - \lambda)w$ for some $w \in H$, $0 < \lambda < 1$. Choose $w' \in K$ such that $\Psi(w') = w$. Then if $x' = \lambda z' + (1 - \lambda)w'$ we have $\Psi(x') = x$ and $x' \in K'$ since $z' \in K'$ and $0 < \lambda < 1$. So $x \in \Psi(K')$.

(2)$\Rightarrow$(1). Suppose $K$ is of dimension $m$ and is in fact contained in $R^m$. If $x \in K'$ then $K$ contains an open line segment containing $x$ in the direction of each coordinate axis. From the convexity of $K$ we deduce that $K$ and hence $K'$ contains an open ball in $R^m$ containing $x$. Hence $K'$ is open in $R^m$.

Choose $\{z_i\}$, a countable dense subset of $E(K)$. Let $\mu = \sum \delta(z_i)/2^i$ ($\delta(z) =$delta measure at $z$). We will show $K' \subset \Phi(\Pi)$ where $\Pi$ is the part of $\mathcal{P}(K)$ containing $\mu$. Choose $y' \in K'$. Let $\Phi(\mu) = x \in K$. Since $y' \in K'$ we can choose $w \in K'$ and $0 < \alpha < 1$ so $y = \alpha z + (1 - \alpha)w$. Choose $\varepsilon > 0$ so, for all $g \in R^m$,

$$\|g - w\| < \varepsilon \Rightarrow g \in K' \quad (\|\cdot\| \text{ is Euclidean norm in } R^m).$$

Choose $n$ so $\{z_1, z_2, \cdots, z_n\}$ is an $\varepsilon$-net for $E(K)$. We claim that $w \in \text{co} \{z_1, z_2, \cdots, z_n\}$. If not $\exists \gamma \in R^n$, $\|\gamma\| = 1$ such that $(\gamma, w) > (\gamma, z_i)$ for $1 \leq i \leq n$. Now $w + \varepsilon \gamma \in K$. Thus $\exists z \in E(K)$ so that

$$(\gamma, z) \geq (\gamma, w + \varepsilon \gamma) = (\gamma, w) + \varepsilon > (\gamma, z_i) + \varepsilon, \quad 1 \leq i \leq n.$$ 

It follows that $\|z - z_i\| > \varepsilon$ if $1 \leq i \leq n$. This contradicts the choice of $n$.

So $w \in \text{co} \{z_1, z_2, \cdots, z_n\}$. This provides a measure $\nu \in \mathcal{P}(K)$ such that $\Phi(\nu) = w$ and $\nu \leq 2^n \mu$. Clearly the probability measure $\alpha \mu + (1 - \alpha)\nu$ represents $y$. It is in $\Pi$ since $\alpha > 0$ and $\nu \leq \alpha \mu + (1 - \alpha)\nu \leq (\alpha + 2^n) \mu$.

**Remarks.** (1) I am grateful to H. S. Bear for his interest in this work. He pointed out to me that my original proof of Theorem 1 was valid only for metrizable $K$, and supplied the simple geometric proof of Lemma 4. I am also grateful to the referee for indicating several places where a few more details would substantially improve the exposition.

(2) A stronger version of Lemma 1 follows immediately from Bauer’s open mapping theorem (to appear in Equaiones Mathematicae, see [3, Theorems 5–13]).
(3) There remains the problem for general parts: Find a condition (geometric or topological) on a part $\Delta$ of $K$ equivalent to (1).

REFERENCES