ON CERTAIN CONVOLUTION INEQUALITIES
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ABSTRACT. It is proved that certain convolution inequalities are easy consequences of the Hardy-Littlewood-Wiener maximal theorem. These inequalities include the Hardy-Littlewood-Sobolev inequality for fractional integrals, its extension by Trudinger, and an interpolation inequality by Adams and Meyers. We also improve a recent extension of Trudinger’s inequality due to Strichartz.

The purpose of this note is to point out that certain convolution inequalities are easy consequences of the Hardy-Littlewood-Wiener maximal theorem. These inequalities include the Hardy-Littlewood-Sobolev inequality for fractional integrals, its extension by Trudinger [11], and an interpolation inequality by Adams and Meyers ([1], [1a]). We also improve a recent extension of Trudinger’s inequality due to Strichartz [10].

Let $f$ be real-valued, Lebesgue measurable, and defined in $\mathbb{R}^d$. For $0<p<\infty$ we write $\|f\|_p = \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx\right)^{1/p}$. For $0<\alpha<d$ the Riesz potentials $I_\alpha(f)$ are defined by $I_\alpha(f)(x) = \int_{\mathbb{R}^d} f(y) |x-y|^{-\alpha} \, dy$. The maximal function $M(f)$ is defined by $M(f)(x) = \sup_{r>0} r^{-d} \int_{|y|<r} |f(x+y)| \, dy$. We will denote various constants, independent of $f$, by $A$.

By the Hardy-Littlewood-Wiener maximal theorem, a simple proof of which is given by Stein [9, I.1],

$$\|M(f)\|_p \leq A\|f\|_p, \quad p > 1.$$  

If $f$ is supported by a finite ball $B$, then (see [9, I.5.2])

$$\int_B M(f) \, dx \leq A \int_B \left(1 + |f| \log^+ |f| \right) \, dx.$$  

The following theorem is due to Hardy and Littlewood [3] for $d=1$, and to Sobolev [8] in the general case. A simple proof is given in [9, V.1.2].

**Theorem 1.** Let $0<\alpha<d$, $1<p<q<\infty$, and $1/q = 1/p - \alpha/d$. Then $\|I_\alpha(f)\|_q \leq A\|f\|_p$. If $f$ is supported by a ball $B$, and $1/q = 1 - \alpha/d$, then $I_\alpha(f) \in L^q(B)$ if $\int_B |f| \log^+ |f| \, dx < \infty$.

We first prove a simple lemma.
Lemma. (a) If $0 < \alpha < d$, then for all $x \in \mathbb{R}^d$ and $\delta > 0$

$$\int_{|y-x| \leq \delta} |f(y)| |x - y|^{a-d} dy \leq A \delta^a M(f)(x).$$

(b) If $\beta > 0$, then

$$\int_{|y-x| \geq \delta} |f(y)| |x - y|^{-\beta-d} dy \leq A \delta^{-\beta} M(f)(x).$$

Proof. We only prove (a), the proof of (b) being similar. For any $x \in \mathbb{R}^d$ and any $\delta > 0$

$$\int_{|y-x| \leq \delta} |f(y)| |x - y|^{a-d} dy = \sum_{n=0}^{\infty} \int_{|y-x| \leq \delta_2^{-n}} |f(y)| |x - y|^{a-d} dy$$

$$\leq A \sum_{n=0}^{\infty} (\delta_2^{-n})^\alpha (\delta_2^{-n})^{-d} \int_{|y-x| \leq \delta_2^{-n}} |f(y)| dy$$

$$\leq A \delta^a M(f)(x) \sum_{n=0}^{\infty} 2^{-na},$$

which proves (a).

Proof of Theorem 1. Let $p \geq 1$, $0 < \alpha < d$, and let $x \in \mathbb{R}^d$ and $\delta > 0$ be arbitrary. By Hölder's inequality for $p > 1$, and immediately for $p = 1$,

$$\int_{|y-x| \geq \delta} |f(y)| |x - y|^{a-d} dy \leq A \|f\|_p \delta^{a-d/p}.$$

Thus, by the lemma

$$|I_\alpha(f)(x)| \leq A(\delta^a M(f)(x) + \delta^{a-d/p} \|f\|_p).$$

To minimize this expression we choose $\delta = (M(f)(x)/\|f\|_p)^{-d/p}$. This gives

$$|I_\alpha(f)(x)| \leq A^2 M(f)(x)^{1-2p/d} \|f\|_p^{2p/d}.$$

The theorem follows immediately from (1) and (2).

Remark. The main difference between the above proof and the proof given in [9] is that the latter depends on the nondiagonal case of the Marcinkiewicz interpolation theorem, whereas the proof of the maximal theorem, as given e.g. in [9], only depends on the easier interpolation along the diagonal, and a covering lemma. On the other hand the proof in [9] is valid for more general kernels.

Theorem 2. Suppose $f \in L^p(\mathbb{R}^d)$ has support in a ball $B$ with diameter $R$, and let $p = d/\alpha > 1$. Then, for any $\varepsilon > 0$ there exists a constant $A_\varepsilon$,
independent of \( f \) and \( R \), such that
\[
\int_{B} \exp \left( \frac{d}{\omega_{d-1}} \left| I_{d}(f)(x) \right| - \epsilon \left| f \right|^{\frac{p}{p-1}} \right) \, dx \leq A_{d} R^{d}.
\]
Here \( \omega_{d-1} \) is the area of the \( d-1 \) dimensional unit sphere.

**Remark.** The above inequality clearly implies that for any \( \beta < d/\omega_{d-1} \) there is an \( A \) so that
\[
\int_{B} \exp \beta \left( \left| I_{d}(f)(x) \right| / \| f \|_{p} \right)^{\frac{p}{p-1}} \, dx \leq A \, R^{d}.
\]

For \( \alpha = 1 \) this implies the inequality of Trudinger [11, p. 479]. In fact, if \( \varphi \) belongs to the Sobolev space \( W_{x}^{2,p}(B) \), then \( |\varphi| \leq \omega_{d-1}^{\frac{1}{p}} \| \text{grad} \, \varphi \| \). (See e.g. [9, V.2.2].) It is known ([4], [6]) that for \( \alpha = 1 \) the inequality does not hold for \( \beta > d/\omega_{d-1} \), but Moser [6] has proved that in this case the inequality is true for \( \beta = d/\omega_{d-1} \), i.e. \( \epsilon \) above can be taken to be zero. The extension of Trudinger’s inequality to \( \alpha \neq 1 \) is due to Strichartz [10], whose proof is also very simple, but does not seem to give quite as sharp a result.

**Proof.** Without loss of generality we can assume that \( \| f \|_{p} = 1 \). As in the proof of Theorem 1 we obtain for any \( x \in B \) and any \( \delta \), \( 0 < \delta \leq R \),
\[
\left| I_{d}(f)(x) \right| \leq A \, \delta^{d} M(f)(x) + (\omega_{d-1} \log(R/\delta))^{1-1/p}.
\]
Now choose \( \delta := \min \{ e^{-1} M(f)(x), R \} \). This gives
\[
\left| I_{d}(f)(x) \right| \leq A \, \delta^{d} M(f)(x) + (\omega_{d-1} \log(R/\delta))^{1-1/p};
\]
and the theorem follows.

The following theorem is a special case of Theorem 4 below, but we prove it separately because of the simplicity of the proof.

**Theorem 3.** Let \( f \geq 0 \) be measurable on \( \mathbb{R}^{d} \). Then
\[
\| I_{d}(f) \|_{r} \leq A \, \| f \|_{p}^{1-\theta} \, \| I_{d}(f) \|_{q}^{\theta},
\]
with \( 0 < \alpha < d \), \( 0 < \theta < 1 \), \( 1 < p < \infty \), \( p < q \leq \infty \), and \( 1/r = (1-\theta)/p + \theta/q \).

**Remark.** For periodic functions, and without the restriction to positive functions, the theorem was proved by Hirschman [5]. For integral \( \alpha \) and \( \alpha \theta \) the theorem follows from the well-known inequalities of Gagliardo [2] and Nirenberg [7]. See also Theorem 4 below.

**Proof.** Let \( x \) be arbitrary and let \( \delta > 0 \). Then clearly
\[
\int_{|y-x| \leq \delta} f(y) \, |x - y|^{\alpha d - d} \, dy \leq \delta^{\alpha d - d} \int_{|y-x| \geq \delta} f(y) \, |x - y|^{\alpha d - d} \, dy \leq \delta^{\alpha d - (\alpha \theta - 1)} I_{d}(f(x)).
\]
Thus, by the lemma \( I_{a\delta}(f)(x) \leq A(\delta^a M(f)(x) + \delta^{a(\theta-1)} I_a(f)(x)) \). Choose
\[ \delta^a = I_a(f)(x)/M(f)(x). \]
It follows that
\[
(5) \quad I_{a\delta}(f)(x) \leq AM(f)(x)^{1-\theta} I_a(f)(x)^\theta.
\]
The theorem follows immediately from Hölder’s inequality and (1).

The following theorem was recently announced by Adams and Meyers [1]. Their proof, which is by complex interpolation, will appear in [1a].

**Theorem 4.** Let \( f \geq 0 \) be measurable on \( \mathbb{R}^d \). Then
\[
\|I_{a\delta}(f^t)\|_r \leq A \|f\|_p^{-\theta} \|I_a(f)\|_q^\theta,
\]
with \( 0 < a < d \), \( 0 < \theta < 1 \), \( 0 < p < \infty \), \( 0 < q \leq \infty \), \( \theta < t < \theta + (1-\theta)p \), and
\[
1/r = (t-\theta)/p + \theta/q.
\]

**Proof.** The case \( t=1 \) was treated above. We first assume \( t>1 \). Then, by the assumptions, \( t<p \). As before, by the lemma,
\[
\int_{|y-x| \leq \delta} f^t(y) \, |x - y|^a dy \leq A \|f\|_p^{-\theta} M(f)(x).
\]
Now choose \( s < p \) so that \( t < \theta + (1-\theta)s \). By Hölder’s inequality
\[
\int_{|y-x| \geq \delta} f^t(y) \, |x - y|^a dy \leq \left( \int_{|y-x| \geq \delta} f(y) \, |x - y|^a dy \right)^{\frac{t}{s-t}} \left( \int_{|y-x| \geq \delta} f^s(y) \, |x - y|^{-a \eta q/(t-1)} dy \right)^{\frac{t-1}{s-1}},
\]
where \( \eta = \theta - t + (1-\theta)s > 0 \).

By part (b) of the lemma
\[
\int_{|y-x| \geq \delta} f^t(y) \, |x - y|^{-a \eta q/(t-1)} dy \leq A \delta^{-\eta q/(t-1)} M(f)(x).
\]
We now observe that \( M(f^t)^{1/t} \leq AM(f^s)^{1/s} \) for \( 0 < t \leq s \). In fact for all
\( a > 0 \), by Hölder's inequality,
\[
\left( \int_{|y-x| < a} f^t(y) \, dy \right)^{1/t} \leq A \left( \int_{|y-x| < a} f^s(y) \, dy \right)^{1/s} \leq AM(f^s)(x)^{1/s}.
\]
Thus
\[
I_{a\delta}(f^s)(x) \leq A \delta^a M(f^s)(x)^{1/s} + A \delta^{-\eta q/(s-1)} M(f^s)(x)^{(t-1)/(s-1)} I_a(f)(x)^{(s-1)/(s-1)}.
\]
To minimize this expression we choose \( \delta^2 = I_A(f)(x)/M(f^\theta)(x)^{1/\theta} \). It follows that

\[
I_{x_0}(f^\theta)(x) \leq AM(f^\theta)(x)^{(1-\theta)/\theta} I_A(f)(x)^\theta.
\]

Since \( \|M(f^\theta)^{1/\theta}\|_p \leq A\|f\|_p \) by (1), the theorem follows by Hölder's inequality.

Now assume \( t < 1 \). We choose \( s_1 \) and \( s_2 \), so that \( 0 < s_2 < s_1 < p \), \( s_1 \leq t \), and so that \( \theta + (1-\theta)s_2 < t < \theta + (1-\theta)s_1 \). By the assumptions this is possible.

Applying Hölder's inequality as in (6) we find

\[
\int_{|y-x| \leq \delta} f(t)(y) |x - y|^{\theta - d} dy \leq \left( \int_{|y-x| \leq \delta} f(y) |x - y|^{\frac{\theta - d}{\theta}} dy \right)^{(1-s_1)/(1-s_1)} \cdot \left( \int_{|y-x| \leq \delta} f^{s_1}(y) |x - y|^{-d + \eta_1/(1-t)} dy \right)^{(1-t)/(1-s_1)},
\]

where \( \eta_1 = \theta - t + (1-\theta)s_1 > 0 \).

Similarly

\[
\int_{|y-x| \geq \delta} f(t)(y) |x - y|^{\theta - d} dy \leq \left( \int_{|y-x| \geq \delta} f(y) |x - y|^{\frac{\theta - d}{\theta}} dy \right)^{(1-s_2)/(1-s_2)} \cdot \left( \int_{|y-x| \geq \delta} f^{s_2}(y) |x - y|^{-d + \eta_2/(1-t)} dy \right)^{(1-t)/(1-s_2)},
\]

where \( \eta_2 = \theta - t + (1-\theta)s_2 < 0 \).

By the lemma

\[
\int_{|y-x| \leq \delta} f^{s_1}(y) |x - y|^{-d + \eta_1/(1-t)} dy \leq A\delta^{\eta_1/(1-t)} M(f^{s_1})(x),
\]

and

\[
\int_{|y-x| \geq \delta} f^{s_2}(y) |x - y|^{-d + \eta_2/(1-t)} dy \leq A\delta^{\eta_2/(1-t)} M(f^{s_2})(x) \leq A\delta^{\eta_2/(1-t)} M(f^{s_2})(x)^{s_2/s_1}.
\]

Thus

\[
I_{x_0}(f^\theta)(x) \leq A\delta^{\eta_1/(1-s_1)} M(f^{s_1})(x)^{(1-t)/(1-s_1)} I_A(f)(x)^{(1-t)/(1-s_1)} + A\delta^{\eta_2/(1-s_2)} M(f^{s_2})(x)^{s_2/(1-s_2)} I_A(f)(x)^{(1-t)/(1-s_2)} \leq A\delta^{\eta_1/(1-s_1)} M(f^{s_1})(x)^{s_1/(1-s_1)} + A\delta^{\eta_2/(1-s_2)} M(f^{s_2})(x)^{s_2/(1-s_2)} I_A(f)(x)^{s_2/(1-s_2)}.\]

By choosing \( \delta^2 = I_A(f)(x)/M(f^{s_1})(x)^{1/s_1} \), we find

\[
I_{x_0}(f^\theta)(x) \leq AM(f^{s_1})(x)^{(1-\theta)/s_1} I_A(f)(x)^\theta,
\]

which proves the theorem.
Note that in the case $t < p$, $t < 1$, we can simplify the proof by choosing $s_1 = t$.

References


1a. ———, Bessel potentials. Inclusion relations among classes of exceptional sets (to appear).


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