ON A CLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. We look at functions \( f(z) \) for which there correspond functions \( \phi(z) \) convex of order \( \alpha \) such that \( \text{Re}\left(\frac{f'(z)}{\phi'(z)}\right) \geq \beta \). We examine the influence of the second coefficient of \( \phi(z) \) on this class. In particular, distortion, covering, and radius of convexity theorems are proved.

1. Introduction. Let \( S \) be the class of normalized univalent functions analytic in the unit disk \( E \) (\( |z| < 1 \)). Let \( K_p(\alpha) \) denote the subclass of \( S \) consisting of functions of the form \( \phi(z) = z + b_2z^2 + \cdots \), where

\[
\text{Re}\left\{z\phi''(z)/\phi'(z) + 1\right\} \geq \alpha, \quad z \in E, \quad |b_2| = p, \quad 0 \leq \alpha \leq 1.
\]

This class is called convex of order \( \alpha \). It is known that \( 0 \leq p \leq 1 \). In addition, \( \phi(z) \in K_p(1) \) if and only if \( \phi(z) = z \) and \( \phi(z) \in K_1(\alpha) \) if and only if \( \phi(z) = z/(1-xz) \), \( |x| = 1 \).

We say that an analytic function \( f(z) = z + a_2z^2 + \cdots \) is in the class \( C_p(\alpha, \beta) \) if there exists a function \( \phi(z) \in K_p(\alpha) \) such that

\[
\text{Re}\left\{f'(z)/\phi'(z)\right\} \geq \beta, \quad \beta \geq 0.
\]

It is easy to verify that for \( \alpha \leq \alpha_0 \) and \( \beta \leq \beta_0 \) we have

\[
C_\alpha(\alpha_0, \beta) \subset C_p(\alpha, \beta) \quad \text{and} \quad C_p(\alpha, \beta_0) \subset C_p(\alpha, \beta).
\]

Kaplan [5] proved that \( C_p(0, 0) \), the class of close-to-convex functions, is univalent. Hence \( C_p(\alpha, \beta) \) is a subclass of \( S \).

By specializing \( \alpha \) and \( \beta \) we obtain some important classes. If \( f(z) \) is in \( C_p(0, 0) \), then \( f(z) \) is close-to-convex; \( C_p(1, \beta) \), then \( \text{Re}\left\{f'(z)/\beta\right\} \geq \beta \); \( C_p(\alpha, 1) \), then \( f(z) \) is convex of order \( \alpha \); \( C_p(1, 1) \), then \( f(z) = z \).

In this note we prove distortion, covering, and radius of convexity theorems for the class \( C_p(\alpha, \beta) \). By specializing \( \beta \), some of our results will coincide with those of Libera [6]. We also look at a corresponding subclass of the close-to-star functions, \( S_p(\alpha, \beta) \). Some of our results for this class will generalize those of Al-Amiri [1] who investigated the class \( S_p(\alpha, 0) \).

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In the sequel, we will assume that \( f(z) \) is in \( C_\rho(\alpha, \beta) \) with \( \phi(z) \) its associated function in \( K_\rho(\alpha) \).

2. **Distortion theorems for \( C_\rho(\alpha, \beta) \).** We begin by proving an existence theorem for functions in this class.

**Theorem 1.** Let \( \alpha \in [0, 1], \beta \in [0, 1], \) and \( p \in [0, 1-\alpha] \). Then there exists a function \( f(z) \in C_\rho(\alpha, \beta) \). This result is sharp in that \( \alpha + p \leq 1 \) for any \( \alpha \).

In proving the theorem we will make use of the following

**Lemma.** Let \( Q(z) \) be analytic for \( z \in E \) with \( Q(0)=1 \). Then \( \Re Q(z) \geq \beta \) if and only if

\[
Q(z) = \frac{1 + (1 - 2\beta)g(z)}{1 - g(z)},
\]

where \( g(z) \) is analytic, \( g(0)=0 \), and \( |g(z)| < 1 \) for \( z \in E \).

**Proof of Lemma.** The result is well known for \( \alpha=0 \). In the general case, let \( g(z) = (1-\alpha)P(z) + \alpha \), where \( P(z) \) satisfies the conditions of the Lemma with \( \alpha=0 \).

**Proof of Theorem 1.** The inequality \( \alpha + p \leq 1 \) is proved in \([1, \text{p. 104}]\).

Let

\[
f(z) = \int_0^z \left( \frac{1}{1-t^2} \right)^{1-z} \left( \frac{1 + t^p}{1 - t} \right) \left( \frac{1 + (1 - 2\beta)t}{1 - t} \right) \, dt
\]

and

\[
\phi(z) = \int_0^z \left( \frac{1}{1-t^2} \right)^{1-z} \left( \frac{1 + t^p}{1 - t} \right) \, dt.
\]

Since

\[
\frac{f'(z)}{\phi'(z)} = \frac{1 + (1 - 2\beta)z}{1 - z}
\]

has real part \( \geq \beta \), it suffices to show that \( \phi(z) \in K_\rho(\alpha) \).

We have

\[
1 + z \frac{\phi''(z)}{\phi'(z)} = \frac{1 + 2pz + (1 - 2\alpha)z^2}{1 - z^2} = \frac{1 + (1 - 2\alpha)g(z)}{1 - g(z)}.
\]

Solving for \( g(z) \), we obtain

\[
g(z) = z \left( \frac{z + p/(1 - \alpha)}{1 + (p/(1 - \alpha))z} \right) = zh(z).
\]

Since \( \alpha + p \leq 1 \), \( h(z) \) maps \( E \to E \), and \( |g(z)| \leq |z| < 1 \) for \( z \in E \). Since \( g(z) \) satisfies the conditions of the Lemma, our proof is complete.
Theorem 2. Let \( f(z) \in C_\nu(\alpha, \beta) \). Then

\[
|f'(z)| \leq \left( \frac{1}{1 - r^2} \right)^{1-\nu} \left( \frac{1 + r}{1 - r} \right)^\nu \left( \frac{1 + (1 - 2\beta)r}{1 - r} \right),
\]

\[
|f'(z)| \geq \frac{1}{1 + (2p/(1 - \alpha))r + r^2^{1-\nu}} \left( \frac{1 - (1 - 2\beta)r}{1 + r} \right),
\]

where the first expression on the right-hand side of (2) is taken to be 1 for \( \alpha = 1 \).

Equality holds in (1) for the function

\[
f_1(z) = \int_0^z \left( \frac{1}{1 - t^2} \right)^{1-\nu} \left( \frac{1 + t}{1 - t} \right)^\nu \left( 1 + (1 - 2\beta)t \right) dt,
\]

and equality holds in (2) for the function

\[
f_2(z) = \int_0^z \frac{1}{1 + (2p/(1 - \alpha))t + r^2^{1-\nu}} \left( \frac{1 - (1 - 2\beta)t}{1 + t} \right) dt.
\]

Proof. From the Lemma we obtain

\[
\frac{f'(z)}{\phi'(z)} = \frac{1 + (1 - 2\beta)g(z)}{1 - g(z)},
\]

where \( g(0) = 0 \) and \( |g(z)| < 1 \) for \( z \in E \).

Since \( g(z) \) satisfies the conditions of Schwarz’s lemma, (3) yields

\[
\frac{1 - (1 - 2\beta)r}{1 + r} \leq \left| \frac{f'(z)}{\phi'(z)} \right| \leq \frac{1 + (1 - 2\beta)r}{1 - r}.
\]

In [1, p. 105] it is proved that

\[
\frac{1}{1 + (2p/(1 - \alpha))r + r^2^{1-\nu}} \leq |f'(z)| \leq \left( \frac{1}{1 - r^2} \right)^{1-\nu} \left( \frac{1 + r}{1 - r} \right)^\nu \left( \frac{1 + (1 - 2\beta)r}{1 - r} \right).
\]

Combining (4) and (5), the result follows. In the proof of Theorem 1 it was shown that \( f_1(z) \in C_\nu(\alpha, \beta) \). The proof that \( f_2(z) \in C_\nu(\alpha, \beta) \) is similar, with

\[
\phi_2(z) = \int_0^z \frac{1}{1 + (2p/(1 - \alpha))t + r^2^{1-\nu}} dt.
\]

Remark. For \( \nu = 1 - \alpha \), (1) reduces to a result of Libera and (2) improves on a result of Libera [6, p. 152]. In his paper, it is claimed that
the function

\[ f(z) = \int_0^z \frac{1 - t}{(1 + i)^{2(1 - \alpha)}(1 + t(1 - 2\beta))} \, dt \]

is in \( C_{1-\alpha}(\alpha, \beta) \) for every \( \alpha \) and \( \beta \). That this is not the case can be seen by letting \( \alpha = 1 \) and \( \beta = \frac{1}{2} \). Then \( f(z) = z - z^2/2 \) and \( \phi(z) = z \). But

\[ \text{Re}\{f'(z)/\phi'(z)\} = \text{Re}\{1 - z\} \]

which is less than \( \frac{1}{2} \) for \( \frac{1}{2} < z < 1 \), and \( f(z) = z - z^2/2 \notin C_{1-\frac{1}{2}}(1, \frac{1}{2}) \).

**Theorem 3.** Let \( f(z) \in C_\mu(\alpha, \beta) \). Then

\[ \int_0^r \frac{1}{(1 + (2p/(1 - \alpha))t + t^2)^{1-\alpha}} \left( \frac{1 - (1 - 2\beta)t}{1 + t} \right) \, dt \leq |f(z)| \]

\[ \leq \int_0^r \left( \frac{1}{1 - t^2} \right)^{1-\alpha} \left( \frac{1}{1 - t} \right)^\mu \left( \frac{1 - (1 - 2\beta)t}{1 + t} \right) \, dt. \]

**Equality holds on the right-hand side for** \( f(z) \) **in Theorem 2 and on the left-hand side for** \( f(z) \) **in Theorem 2.**

**Proof.** Integrating along the straight line segment from the origin to \( z = re^{i\theta} \) and applying Theorem 2 we obtain

\[ |f(z)| \leq \int_0^r |f'(te^{i\theta})| \, dt \leq \int_0^r \left( \frac{1}{1 - t^2} \right)^{1-\alpha} \left( \frac{1}{1 - t} \right)^\mu \left( \frac{1 - (1 - 2\beta)t}{1 + t} \right) \, dt, \]

which proves the right-hand inequality. To prove the left-hand inequality, for every \( r \) we choose \( z_0, |z_0| = r \), such that

\[ |f(z_0)| = \min_{|z|=r} |f(z)|. \]

If \( L(z_0) \) is the pre-image of the segment \( \{0, f(z_0)\} \), then

\[ |f(z)| \geq |f(z_0)| \geq \int_{L(z_0)} |f'(z)| \, |dz| \]

\[ \geq \int_0^r \frac{1}{(1 + (2p/(1 - \alpha))t + t^2)^{1-\alpha}} \left( \frac{1 - (1 - 2\beta)t}{1 + t} \right) \, dt \]

and this completes the proof.

For \( p = 1 - \alpha \), the right-hand inequality reduces to a result of Libera and the left-hand inequality sharpens a result of Libera [6, p. 152].

3. **Covering theorems for** \( C_\mu(\alpha, \beta) \). We first prove a coefficient theorem for the class.
Theorem 4. Let \( f(z) \in C_p(\alpha, \beta) \), with \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \). Then \( |a_2| \leq 1 + p - \beta \), with extremal function

\[
f(z) = \int_0^z \left( \frac{1}{1 - t^2} \right)^{1-a} \left( \frac{1 - (1 - 2\beta)t}{1 - t} \right) dt.
\]

Proof. Let \( \phi(z) = z + \sum_{n=2}^{\infty} b_n z^n \). Then

\[
g(z) = \frac{f'(z)/\phi'(z) - \beta}{1 - \beta} = 1 + \frac{2(a_2 - b_2)}{1 - \beta} + \sum_{n=3}^{\infty} c_n z^n
\]

has positive real part in \( E \). Hence [3, p. 15], \( 2|a_2 - b_2|/(1 - \beta) \leq 2 \), or \( |a_2| \leq 1 + |b_2| - \beta = 1 + p - \beta \).

Theorem 5. Let \( f(z) \in C_p(\alpha, \beta) \), with \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \). If \( f(z) \neq k \) for \( z \in E \), then \( |k| \geq 1/(3 + p - \beta) \).

Proof. If \( f(z) \) does not assume the value \( k \), then

\[
\frac{kf(z)}{k - f(z)} = z + (a_2 + 1/k)z^2 + \sum_{n=3}^{\infty} c_n z^n
\]

is in the class \( S \). Hence,

\[
|a_2 + 1/k| \leq 2.
\]

Applying the triangle inequality and Theorem 4 to (6) we obtain Theorem 5.

Since \( p \leq 1 - \alpha \), we also obtain the following result of Libera [6, p. 155] as a

Corollary. \( |k| \geq 1/(4 - \alpha - \beta) \).

4. A radius of convexity theorem for \( C_p(\alpha, \beta) \).

Theorem 6. Let \( f(z) \in K_p(\alpha, \beta) \). Then \( f(z) \) maps the disk \( |z| < R \) onto a convex domain, where \( R \) is the least positive root of the equation \( a(r, p, \alpha, \beta) = 0 \), where

\[
a(r, p, \alpha, \beta) = (1 - \alpha)(1 - 2\alpha)(1 - 2\beta)r^4
\]

\[
-2[(1 - \alpha)(1 - \beta) + \alpha p(1 - 2\beta)]r^3
\]

\[
-2[(1 - \alpha)(1 - \alpha - \beta - 2p\beta) + 2p^2]r^2
\]

\[
-2[1 - \alpha(1 + p)]r + (1 - \alpha).
\]

Proof. Let \( f'(z)/\phi'(z) = Q(z) \), where \( \Re Q(z) \geq \beta \). Since

\[
\frac{f''(z)}{f'(z)} = \frac{\phi''(z)}{\phi'(z)} + \frac{Q'(z)}{Q(z)},
\]

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the radius of convexity of \( f(z) \) is at least equal to the smallest positive root of
\[
1 + \min \operatorname{Re}\{zf''(z)/f'(z)\} + \min \operatorname{Re}\{zQ'(z)/Q(z)\} = 0.
\]

In \([1, \text{p. 105}]\), it is shown that
\[
\operatorname{Re}\left\{ \frac{z f''(z)}{f'(z)} \right\} \geq - \frac{2r(1 - \alpha)[r(1 - \alpha) + p]}{(1 - \alpha)(1 + r^2) + 2pr}.
\]

Now let \( Q(z) = (1 - \beta)P(z) + \beta \), where \( P(z) \) is analytic, \( P(0) = 1 \), and \( \operatorname{Re} P(z) > 0 \) in \( E \). Then
\[
\frac{zQ'(z)}{Q(z)} = \frac{(1 - \beta)P'(z)}{(1 - \beta)P(z) + \beta} = \frac{P'(z)}{P(z) + \beta(1 - \beta)}.
\]

Using a lemma of Libera \([6, \text{p. 150}]\) we obtain
\[
\left| \frac{zQ'(z)}{Q(z)} \right| \leq \frac{2r}{(1 - r)[1 + r + (\beta/(1 - \beta))(1 - r)]}.
\]

Substituting (8) and (9) into (7) yields
\[
\operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq 1 - \frac{2r(1 - \alpha)[r(1 - \alpha) + p]}{(1 - \alpha)(1 + r^2) + 2pr}
\]
\[
- \frac{2r}{(1 - r)[1 + r + (\beta/(1 - \beta))(1 - r)]}.
\]

Simplifying the right-hand side of (10) we obtain
\[
\frac{a(r, \alpha, \beta)}{[(1 - \alpha)(1 + r^2) + 2pr][(1 - r)(1 + r + (\beta/(1 - \beta))(1 - r)]},
\]
and this completes the proof.

\textbf{Remark 1}. The Koebe function is in \( C_1(0, 0) \), and the least positive root of
\[
a(r, 1, 0, 0) = r^4 - 2r^3 - 6r^2 - 2r + 1
\]
is \( 2 - \sqrt{3} \), the radius of convexity for the class \( S \).

\textbf{Remark 2}. If \( p = 1 - \alpha \),
\[
a(r, 1 - \alpha, \alpha, \beta) = (r + 1)(1 - \alpha)[(1 - 2\alpha)(1 - 2\beta)r^3
\]
\[
- (3 - 6\beta + 4\alpha\beta)r^2 + (2\alpha - 3)r + 1].
\]

This reduces to a result of Libera \([6, \text{p. 151}]\).
5. The class $S_p(\alpha, \beta)$. Let $S_p^*(\alpha)$ denote the class of functions analytic in $E$ and of the form
$s(z) = z + a_2z^2 + \cdots$, where $s(z)$ is starlike of order $\alpha$ and $|a_2| = 2\beta$.

Let $S_p(\alpha, \beta)$ denote the class of functions analytic in $E$ and of the form
$g(z) = z + b_2z^2 + \cdots$ such that $\text{Re}\{g(z)/s(z)\} \geq \beta$ for $z \in E$ and for some $s(z) \in S_p^*(\alpha)$.

The class $S_p(0, 0)$, defined by Reade [7], is called close-to-star. It is known that members of this class need not be univalent. However, there is an important connection between the classes $S_p(\alpha, \beta)$ and $C_p(\alpha, \beta)$ which we state as

**Theorem 7.** The following relationships hold:

(11) \[ f(z) \in C_p(\alpha, \beta) \text{ if and only if } zf'(z) \in S_p(\alpha, \beta), \]

(12) \[ f(z) \in S_p(\alpha, \beta) \text{ if and only if } \int_0^z \frac{f(t)}{t} \, dt \in C_p(\alpha, \beta). \]

**Proof.** It is well known that $\phi(z)$ is convex of order $\alpha$ if and only if $z\phi'(z)$ is starlike of order $\alpha$. Hence, $\phi(z) = z + a_2z^2 + \cdots$ is in $K_p(\alpha)$ if and only if $z\phi'(z) = z + 2a_2z^2 + \cdots$ is in $S_p^*(\alpha)$. Since

\[ \text{Re}\{f'(z)/\phi'(z)\} = \text{Re}\{zf'(z)/z\phi'(z)\}, \]

we obtain (11). The proof of (12) is similar and will be omitted.

**Theorem 8.** Let $g(z) \in S_p(\alpha, \beta)$. Then

\[ \frac{r}{(1 + (2\beta/1 - \alpha)r + r^2)^{1-\alpha}} \left(\frac{1 - (1 - 2\beta)r}{1 + r}\right) \leq |g(z)| \leq \frac{r}{(1 - r)^{1-\alpha}} \left(\frac{1 + r}{1 - r}\right)^\beta \left(\frac{1 + (1 - 2\beta)r}{1 + (2\beta/1 - \alpha)r + r^2}\right). \]

**Proof.** The result follows on combining Theorem 2 with Theorem 7.

**Corollary.** Let $g(z)$ be analytic in $E$ with $\text{Re}\{g(z)/z\} > \frac{1}{2}$. Then

(13) \[ r/(1 + r) \leq |g(z)| \leq r/(1 - r). \]

**Proof.** We have $\text{Re}\{g(z)/z\} > \frac{1}{2}$ if and only if $g(z) \in S_0(1, \frac{1}{2})$. Schild [8, p. 752] proved (13) for the class of functions starlike of order $\frac{1}{2}$, a subclass of $S_0(1, \frac{1}{2})$ [4, p. 472].

For a functional analysis proof of the corollary, see [2, p. 94].

Once again making use of Theorem 7, we see that $g(z) \in S_p(\alpha, \beta)$ if and only if

\[ \frac{z \frac{g'(z)}{g(z)} = 1 + \frac{z \frac{f''(z)}{f'(z)}}{f'(z)}}{f''(z)} \]
for some \( f(z) \in C_p(\alpha, \beta) \). Hence a radius of convexity theorem in \( C_p(\alpha, \beta) \) will correspond to a radius of starlikeness theorem in \( S_p(\alpha, \beta) \). From Theorem 6 we now obtain

**Theorem 9.** Let \( g(z) \in S_p(\alpha, \beta) \). Then \( g(z) \) maps the disk \( |z| < R \) onto a starlike domain, where \( R \) is the least positive root of \( a(r, p, \alpha, \beta) = 0 \), defined in Theorem 6.

For \( \beta = 0 \), this reduces to a result of Al-Amiri [1, p. 108].

**Bibliography**


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