A SPLITTING THEOREM FOR ALGEBRAS OVER COMMUTATIVE VON NEUMANN REGULAR RINGS

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Abstract. Let $R$ be a commutative von Neumann ring. Let $A$ be an $R$-algebra which is finitely generated as an $R$-module and has $A/N$ separable over $R$. Here $N$ is the Jacobson radical of $A$. Then it is shown that there exists an $R$-separable subalgebra $S$ of $A$ such that $S+N=A$ and $S\cap N=0$. Further it is shown that if $T$ is another $R$-separable subalgebra of $A$ for which $T+N=A$ and $T\cap N=0$, then there exists an element $n\in N$ such that $(1-n)S(1-n)^{-1}=T$. This result is then used to determine the structure of all strong inertial coefficient rings.

Introduction. The purpose of this note is to prove the following theorem: Let $R$ be a commutative von Neumann regular ring. Let $A$ be an $R$-algebra which is finitely generated as an $R$-module and has $A/N$ separable over $R$, $N$ the Jacobson radical of $A$. Then there exists a separable $R$-subalgebra $S$ of $A$ such that $S+N=A$ and $S\cap N=0$. In terms of the definitions of [1], this theorem states that the pair $(R, 1)$ is a strong inertial coefficient ring if $R$ is a von Neumann regular ring. This theorem is then used to obtain a complete characterization of all strong inertial coefficient rings.

Preliminaries. Throughout this paper, any ring will be assumed to be associative and to contain an identity element. All subrings of a given ring are assumed to contain the identity of the given ring. All ring homomorphisms are assumed to take the identity to identity. $R$ will always denote a commutative ring and $A$ an $R$-algebra. We shall let $p$ and $N$ denote the Jacobson radicals of $R$ and $A$ respectively.

Let $\pi_0: R\rightarrow R/p$ be the natural projection of $R$ onto $R/p$. Then $R$ together with a ring homomorphism $\delta: R/p\rightarrow R$ will be called a pair and written $(R, \delta)$ if $\pi_0\delta$ is the identity map on $R/p$. The pair $(R, \delta)$ is called a strong inertial coefficient ring if for every $R$-algebra $A$ which is finitely generated as an $R$-module and has $A/N$ separable over $R$, there exists an $(R/p)$-separable subalgebra $S$ of $A$ such that $S+N=A$ and $S\cap N=0$. The basic properties of strong inertial coefficient rings can be found in [1].

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The ring $R$ is a von Neumann regular ring if for every $z$ in $R$ there exists a $y$ in $R$ such that $zyz = z$. The proof of the main result in this paper is based on the theorems and techniques which appear in [6]. The author assumes the reader is familiar with these results.

**The main result.** We begin with the following lemma:

**Lemma 1.** Let $A$ be an $R$-algebra. Then $A$ is separable over $R$ if there exist elements ${a_1, \ldots, a_n}$ and $\{a'_1, \ldots, a'_n\}$ in $A$ such that

1. $\sum_{i=1}^{n} a_i a'_i = 1$;
2. for every $a \in A$, there exist constants $\lambda_{ij}(a) \in R$, $i, j = 1, \ldots, n$ such that

$$a_i a = \sum_{j=1}^{n} \lambda_{ij} a_j \quad \text{and} \quad a a'_i = \sum_{j=1}^{n} \lambda_{ij} a'_j.$$

**Proof.** Let $I$ denote the kernel of the multiplication mapping $\mu: A \otimes_R A^o \to A$. Then $A$ is separable over $R$ if and only if $0 \to I \to A \otimes_R A^o \to A \to 0$ splits as $(A \otimes_R A^o)$-modules. That is, $A$ is separable over $R$ if and only if there exists an $(A \otimes_R A^o)$-module homomorphism $P: A \to A \otimes_R A^o$ such that $\mu P$ is the identity map on $A$. Conditions 1° and 2° of the lemma imply that the map $P: A \to A \otimes_R A^o$ defined by $P(1) = \sum_{i=1}^{n} a_i \otimes a_i$ is an $(A \otimes_R A^o)$-module homomorphism for which $\mu P$ is the identity map. \(\square\)

**Theorem 1.** Let $R$ be a commutative von Neumann regular ring. Let $A$ be an $R$-algebra which is finitely generated as an $R$-module and has $A/N$ separable over $R$. Then there exists an $R$-separable subalgebra $S$ of $A$ such that $S + N = A$ and $S \cap N = 0$.

**Proof.** Let $X(R)$ denote the decomposition space of $R$ [6, p. 8]. Let $x \in X(R)$. If $M$ is any $R$-module, we shall let $M_x$ be $M \otimes_R R/xR$. It is well known that $\otimes_R R_x$ is an exact functor. Hence $0 \to N_x \to A_x \to (A/N)_x \to 0$ is an exact sequence of $R_x$-algebras. Since $R$ is a von Neumann regular ring, $R_x$ is a field. $(A/N)_x$ being a homomorphic image of $A/N$ is separable over $R_x$. Thus $N_x$ is the Jacobson radical of $A_x$. By Wedderburn’s theorem, there exists an $R_x$-subalgebra $S_x$ of $A_x$ such that $N_x \otimes S_x = A_x$.

If $a \in A$, we shall denote by $a_x$ the image of $a$ in $A_x = A/xA$. Thus $a_x = a + xA$ ($x = xR$). Now let $\{a_1, \ldots, a_n\}$ be a set of $R$-module generators for $A$. Then if $\pi: A \to A/N$ denotes the natural projection of $A$ onto $A/N$, we have $\{(a_1)_x, \ldots, (a_n)_x\}$ generates $A_x$ as an $R_x$-module, $\{\pi(a_1) = \tilde{a}_1, \pi(a_n) = \tilde{a}_n\}$ generates $A/N$ as an $R$-module and $\{(\tilde{a}_1)_x, \ldots, (\tilde{a}_n)_x\}$ generates $(A/N)_x$ as an $R_x$-module. Since $S_x$ is an $R_x$-separable subalgebra of $A_x$, which is isomorphic to $(A/N)_x$, we can make the following statements:
1°. There exist elements $s_1^x, \ldots, s_{m(x)}^x \in S_x$ and elements $r_{ijk}^x \in R_x, i, j, k = 1, \ldots, m(x)$ and $r_i^x \in R_x, i = 1, \ldots, m(x)$ such that

(a) $\{s_1^x, \ldots, s_{m(x)}^x\}$ is a vector space basis of $S_x$ over $R_x$,
(b) $s_is_j^x = \sum_{k=1}^{m(x)} r_{ijk}^x s_k^x$ for all $i, j = 1, \ldots, m(x)$,
(c) $1_x = \sum_{i=1}^{m(x)} r_i^x s_i^x$.

Here $m(x)$ denotes some positive integer depending on $x$.

2° [4, Theorem 7.6]. There exist elements $r_i^x \in R_x, i = 1, \ldots, m(x)$ such that $s_i^x = \sum_{j=1}^{m(x)} r_{ij}s_j^x$, for $i = 1, \ldots, m(x)$, satisfy the following two properties:

(a) $\sum_{j=1}^{m(x)} s_j^x s_j^x = 1_x$,
(b) for all $s^x \in S_x$, if $s_i^x s^x = \sum_{i=1}^{m(x)} \lambda_{ij}s_j^x$ for $i = 1, \ldots, m(x)$ ($\lambda_{ij} \in R_x$), then $s^x s_i^x = \sum_{j=1}^{m(x)} \lambda_{ij}s_j^x$.

3°. There exist elements $t_{ij}^x \in R_x, i = 1, \ldots, n, j = 1, \ldots, m(x)$, and elements $z_i^x \in N_x$ such that

$$a_i - \sum_{j=1}^{m(x)} t_{ij}s_j^x z_i = 0$$

for $i = 1, \ldots, n$.

Now let $\{r_{ijk}, r_i^x\}$ and $\{t_{ij}\} \in R$ such that their images in $R_x$ are $\{r_{ijk}^x, r_i^x\}$ and $\{t_{ij}^x\}$ respectively. Similarly let $\{s_1^x, \ldots, s_{m(x)}^x\}$ and $\{z_1^x, \ldots, z_n^x\}$ be elements in $A$ and $N$ respectively such that their images are $\{s_1^x, \ldots, s_{m(x)}^x\}$ and $\{z_1^x, \ldots, z_n^x\}$ in $A_x$. Now the elements

$$s_j s_j^x = - \sum_{k=1}^{m(x)} r_{ijk}s_k^x, \quad 1 - \sum_{i=1}^{m(x)} r_i^x s_i^x \quad \text{and} \quad a_i - \sum_{j=1}^{m(x)} t_{ij}s_j^x = z_i^x$$

may be viewed as global sections on the sheaf $\mathcal{A}(A)$ [6, p. 18] over $X(R)$. These sections are zero at $x$ and hence are zero on some open set $U$ of $X(R)$ containing $x$. Thus for all $y \in U$, the sections $s_1^x, \ldots, s_{m(x)}^x$ generate an $R_x$-subalgebra $S_y$ of $A_y$ such that $N_y + S_y = A_y$.

Now $s_i^x = \sum_{j=1}^{m(x)} r_{ij}s_j^x, i = 1, \ldots, m(x)$, may also be viewed as sections on $\mathcal{A}(A)$. The section $\sum_{i=1}^{m(x)} (\sum_{j=1}^{m(x)} r_{ij}s_j) s_i - 1$ is zero at $x$. Hence by shrinking $U$ if need be, we may assume for all $y \in U$,

$$\sum_{i=1}^{m(x)} (s_i^y) s_i^y = 1_y.$$

By 1°(b) and 2°(b), we have $s^x s^x = \sum_{k=1}^{m(x)} r_{kij}s_k^x$ for all $i, j = 1, \ldots, m(x)$.

Thus, by shrinking $U$ further if need be, we can assume for all $y \in U$ and for all $i, j = 1, \ldots, m(x)$,

$$(s_i^y) s_i^y = \sum_{k=1}^{m(x)} (r_{kij}) y (s_k^y).$$
Since each $S_y$, for $y \in U$, is generated as an $R_y$-module by $(s_1)_y, \ldots, (s_{m(x)})_y$, we get for every $a \in S_y$ there exist constants $\lambda_{ij}(a)$ in $R_y$ such that

$$(s_j)_y a = \sum_{j=1}^{m(x)} \lambda_{ij}(a)(s_j)_y \quad \text{and} \quad a(s_j)_y = \sum_{j=1}^{m(x)} \lambda_{ij}(a)(s_j)_y.$$ 

Thus by Lemma 1, each $S_y$ for $y \in U$ is a separable $R_y$-subalgebra of $A_y$. Since $R_y$ is a field, $S_y$ is semisimple. Hence $N_y \cap S_y = 0$.

Since $x$ was an arbitrary point of $X(R)$, we have proven the following assertion: For each point $x$ in $X(R)$, there exists an open set $U_x$ in $X(R)$ containing $x$ and there exist elements $s_1(x), \ldots, s_{m(x)}(x), s'_1(x), \ldots, s'_{m(x)}(x) \in A$, $z_1(x), \ldots, z_n(x) \in N$, and elements $\{r_{ij,k}(x)\}, \{r_i(x)\}, \{r_l(x)\}, \{t_{ij}(x)\} \in R$ such that the elements $(s_1)_y, \ldots, (s_{m(x)})_y$, generate an $R_y$-subalgebra $S_y$ for which $S_y \oplus N_y = A_y$, for all $y \in U_x$. Now $\{U_x : x \in X(R)\}$ is an open covering of $X(R)$. Hence by the partition property, there exist a finite number of open and closed, pairwise disjoint subsets $N_1, \ldots, N_q$ of $X(R)$ such that $\bigcup N_i = X(R)$ and each $N_i$ is contained in some $U_x$.

Let $x_1, \ldots, x_q$ be elements in $X(R)$ such that $N_i \subset U_{x_i}$ for $i = 1, \ldots, q$. On each $N_i$ we may restrict the sections $s_1(x_i), \ldots, s_{m(x_i)}(x_i), s'_1(x_i), \ldots, s'_{m(x_i)}(x_i), z_1(x_i), \ldots, z_n(x_i), \{r_{ij,k}(x_i)\}$, etc. Let $m = \max\{m(x_1), \ldots, m(x_q)\}$. Since the $N_i$'s are pairwise disjoint, we may piece the sections together on each open set $N_i$ to form global sections

$$\tilde{s}_1, \ldots, \tilde{s}_m, \tilde{s}'_1, \ldots, \tilde{s}'_m \in \Gamma(X(R), \mathcal{A}(A)), \quad \tilde{z}_1, \ldots, \tilde{z}_n \in \Gamma(X(R), \mathcal{A}(N))$$

and $\{\tilde{r}_{ij,k}\}, \{\tilde{r}_i\}, \{\tilde{r}_l\}, \{\tilde{t}_{ij}\} \in \Gamma(X(R), \mathcal{B}(R))$ as follows:

For $x \in N_i, i = 1, \ldots, q$, \quad \tilde{s}_j(x) = \begin{cases} s_j(x_i)_x & \text{if } 1 \leq j \leq m(x_i), \\ 0 & \text{if } j > m(x_i). \end{cases}$

The other sections are defined similarly. We now have for each $x \in X(R)$, $\{\tilde{s}_1(x), \ldots, \tilde{s}_m(x)\}$ generates an $R_x$-subalgebra $S_x$ of $A_x$ for which $N_x \oplus S_x = A_x$.

Now by [6, Theorem 4.4 and Theorem 4.5], $\Gamma(X(R), \mathcal{A}(A)) \cong A$, $\Gamma(X(R), \mathcal{A}(N)) \cong N$ and $\Gamma(X(R), \mathcal{B}(R)) \cong R$. Thus there exist elements $\tilde{s}_1, \ldots, \tilde{s}_m \in A, \tilde{z}_1, \ldots, \tilde{z}_n \in N$ and elements $\{\tilde{r}_{ij,k}\}, \{\tilde{r}_i\}, \{\tilde{r}_l\}, \{\tilde{t}_{ij}\}$ in $R$ such that for every $x$ in $X(R)$

$$\tilde{s}_i(x) = (\tilde{s}_i)_x = \tilde{s}_i + \tilde{x}A,$$

$$\tilde{z}_i(x) = (\tilde{z}_i)_x = z_i + \tilde{x}N,$$

$$\tilde{r}_{ij,k}(x) = (\tilde{r}_{ij,k})_x = \tilde{r}_{ij,k} + xR, \quad \text{etc.}$$

Since $\bigcap_{x \in X(R)} \tilde{x}A = 0$, it follows easily that $S = \sum_{i=1}^{m} \tilde{s}_i R$ is an $R$-subalgebra of $A$ such that $S + N = A$ and $S \cap N = 0$. Since $S$ is isomorphic to $A/N$, $S$ is separable. $\Box$
Corollary. Let $R$ have Jacobson radical zero. Then the pair $(R, 1)$ is a strong inertial coefficient ring if and only if $R$ is a von Neumann regular ring.

Proof. By Theorem 1, if $R$ is a von Neumann regular ring, then $(R, 1)$ is a strong inertial coefficient ring. It follows from the proof of [3, Proposition 1] (whether $R$ is assumed Noetherian or not) that if $(R, 1)$ is a strong inertial coefficient ring, then $I=I^2$ for every ideal $I$ in $R$. Hence if $z \in R$, then $zR=(zR)^2$. So there exists a $y \in R$ such that $zyz=z$. □

The Malcev analog of Theorem 1 follows immediately from [5, Corollary 2.4]. Thus under the hypotheses of Theorem 1, if $S$ and $T$ are two separable $R$-subalgebras of $A$ such that $S \oplus N=A$ and $T \oplus N=A$, then there exists an element $n \in N$ such that $(1-n)S(1-n)^{-1}=T$.

In terms of the definitions in [1], Theorems 1 and 2 may be summarized as follows: If $R$ is a von Neumann regular ring, then $(R, 1)$ is a strong inertial coefficient ring with the uniqueness property.

In [2], the author and E. Ingraham completely characterized all semi-local inertial coefficient rings. Namely, a ring $R$ is an inertial coefficient ring with finitely many maximal ideals if and only if $R$ is a finite direct sum of Hensel rings. If $(R, \mathcal{E})$ is a strong inertial coefficient ring, then $R$ is an inertial coefficient ring [1, Proposition 1]. Thus using the previous result, we get $(R, \mathcal{E})$ is a strong inertial coefficient ring with finitely many maximal ideals if and only if $R$ is a finite direct sum of split Hensel rings. In this paper, we have determined the structure of all (Jacobson) semi-simple strong inertial coefficient rings. We may use these two results to give a complete characterization of strong inertial coefficient rings.

Theorem 2. A pair $(R, \mathcal{E})$ is a strong inertial coefficient ring if and only if for every $x \in X(R)$, $R_x=R/xR$ is a Hensel ring.

Proof. Suppose that for each $x$ in $X(R)$, $R_x$ is a Hensel ring. Then $(R_x, \mathcal{E}_x)$ is a strong inertial coefficient ring. Thus the same proof as used in Theorem 1 with minor changes shows that $(R, \mathcal{E})$ is a strong inertial coefficient ring.

Conversely, for any pair $(R, \mathcal{E})$ we note that $X(R)=X(\mathcal{E}(R/p))$. If we assume $(R, \mathcal{E})$ is a strong inertial coefficient ring, then for any $x \in X(R)$ the pairs $(R_x, \mathcal{E}_x)$ and $(R/p, 1)$ are also strong inertial coefficient rings. By the corollary to Theorem 1, $R/p$ is a von Neumann regular ring. Now

$$0 \rightarrow p_x \rightarrow R_x \rightarrow (R/p)_x \rightarrow 0$$

is exact and $(R/p)_x=(R/p)/x(R/p)$ is a field. Thus $R_x$ is a quasilocal ring. It now follows from [2, Theorem] that $R_x$ is a Hensel ring.
References


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