TOEPLITZ OPERATORS AND DIFFERENTIAL EQUATIONS ON A HALF-LINE\textsuperscript{1}

J. W. MOELLER

Abstract. Let $\mathcal{H}$ be a separable Hilbert space, let $A_0, A_1, \cdots, A_n$ denote bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$, and let $\mathcal{D}$ represent the set of all functions in $L^2(0, \infty; \mathcal{H})$ whose first $n$ derivatives belong to $L^2(0, \infty; \mathcal{H})$. Suppose further that the space $\mathcal{D}$ is equipped with an inner product inherited from $L^2(0, \infty; \mathcal{H})$.

The main result of this note states that the differential operator

$$L = \frac{d^n}{dt^n} + \frac{d^{n-1}}{dt^{n-1}} + \cdots + A_1 \frac{d}{dt} + A_0$$

acting on $\mathcal{D}$ is continuously invertible if and only if the operator

$$P(\sigma) = \sum A_k^* \sigma^k \quad (0 \leq k \leq n)$$

acting on the Hilbert space $\mathcal{H}$ has a uniformly bounded inverse everywhere in the open half-plane $\text{Re} \sigma < 0$.

Let $\mathcal{H}$ be a separable Hilbert space, and let $A_0, A_1, \cdots, A_n$ denote bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$. In what follows we will obtain necessary and sufficient conditions to insure the continuous invertibility of the differential operator

$$L = \frac{d^n}{dt^n} + \frac{d^{n-1}}{dt^{n-1}} + \cdots + A_1 \frac{d}{dt} + A_0$$

acting on a dense manifold of the Hilbert space $L^2(0, \infty; \mathcal{H})$.

Our approach to this problem is based on the observation that $L$ is unitarily equivalent to a generalized Toeplitz operator. The inversion theory of these operators was recently developed by Rabindranathan [5], who systematically extended the previous work of Widom [7], Devinatz [1], and Pousson [4]. Hereafter we will freely use the terminology, as well as some of the theory, contained in Rabindranathan's paper.

To expose the connections between $L$ and a generalized Toeplitz operator, we first construct a special isometric mapping from $L^2(0, \infty; \mathcal{H})$. 

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onto $H^2(\mathcal{H})$, the Hardy space of $\mathcal{H}$-valued analytic functions defined on the interior of the unit disc. This is accomplished by taking the Laguerre functions

$$g_n(t) = \frac{1}{n!} \exp \left( \frac{t}{2} \right) \frac{d^n}{dx^n} \left[ t^n \exp(-t) \right]$$

and then defining $J(g_n \varphi) = z^n \varphi$, $n = 0, 1, \ldots$, for all $\varphi \in \mathcal{H}$. Since the Laguerre functions constitute an orthonormal basis for $L^2(0, \infty)$, the map $J$ may be extended linearly as an isometry from $L^2(0, \infty; \mathcal{H})$ onto $H^2(\mathcal{H})$. In the scalar case where $\dim \mathcal{H} = 1$, this mapping was employed by the author to study differentiability properties of exponential sums [3], and it also occurs in Rosenblum’s earlier work on selfadjoint Toeplitz operators [6].

From the definition of a Laguerre function we deduce that

$$-2 \frac{dg_n}{dt}(t) = g_n(t) + 2 \sum_{k=0}^{n-1} g_k(t) \quad (0 \leq k \leq n - 1),$$

and a short computation reveals the important identity

$$(2) \quad 2JDN^{-1} = (T + I)(T - I)^{-1},$$

where

$$(Tf)(z) = z^{-1}(f(z) - f(0))$$

and

$$(Dg)(t) = \lim_{h \to 0} h^{-1}(g(t + h) - g(t)).$$

This last limit is taken with respect to the norm topology on $L^2(0, \infty; \mathcal{H})$. Since the open unit disc comprises the point spectrum of $T$, the right side of (2) is well defined on the range of $T - I$, a set whose closure is all of $H^2(\mathcal{H})$ because it contains every vector-valued polynomial

$$p(z) = \varphi_n z^n + \varphi_{n-1} z^{n-1} + \cdots + \varphi_1 z + \varphi_0$$

with coefficients in $\mathcal{H}$.

If $A$ is an operator from $\mathcal{H}$ into $\mathcal{H}$, we designate its natural extension $\hat{A}$ on $H^2(\mathcal{H})$ by writing

$$(\hat{A}f)(z) = \sum (A \varphi_n) z^n, \quad n = 0, 1, 2, \ldots,$$

whenever $f(z) = \sum \varphi_n z^n$. Clearly $\hat{A}$ commutes with $T$, and the substitution of (2) into (1) yields

$$(3) \quad JLJ^{-1} = \sum 2^{-k} \hat{A}_k (T + I)^k (T - I)^{-k} \quad (0 \leq k \leq n).$$
We infer from (3) that $L$ has a bounded inverse if and only if the operator
\[ S = (T - I)^{-n} \sum 2^{-k} \hat{A}_k(T + I)^k(T - I)^{n-k} \quad (0 \leq k \leq n) \]
shares this property too.

A more effective method for determining the continuous invertibility of $S$ may be obtained by examining the adjoint operator $S^*$. Since
\[ (T^*f)(z) = zf(z) \quad \text{and} \quad (\hat{A}^*f)(z) = \sum (A^*\varphi_n)z^n, \]
a standard calculation involving the adjoint of a densely defined operator [2, p. 69] shows that
\[ (S^*f)(z) = R(z)f(z) \]
where
\[ R(z) = \sum 2^{-k} A^*_k(z + 1)^k(z - 1)^{-k} \quad (0 \leq k \leq n). \]

According to a well-known result (Lemma 4.2 in [5]), $S^*$ has a bounded inverse if and only if there exists an analytic Toeplitz operator $Q(z) = \sum Q_n z^n$, $n = 0, 1, \cdots$, defined on the interior of the unit disc such that
\[ R(z)Q(z) = Q(z)R(z) = I \quad \text{and} \quad \sup_{|z|<1} \|Q(z)\| < \infty. \]

With this information at hand, it is possible to enunciate a simple invertibility criterion.

**Theorem.** Let $\mathcal{H}$ be a separable Hilbert space, let $A_0, A_1, \cdots, A_n$ denote bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$, and let $\mathcal{D}$ represent the set of all functions in $L^2(0, \infty; \mathcal{H})$ whose first $n$ derivatives lie in $L^2(0, \infty; \mathcal{H})$. Suppose further that $\mathcal{D}$ is endowed with the inner product inherited from $L^2(0, \infty; \mathcal{H})$. Then the differential operator
\[ L = A_n \frac{d^n}{dt^n} + A_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \cdots + A_1 \frac{d}{dt} + A_0 \]
acting on $\mathcal{D}$ has a bounded inverse if and only if the operator
\[ P(\sigma) = \sum A_k^* \sigma^k \quad (0 \leq k \leq n) \]
acting on the Hilbert space $\mathcal{H}$ has a uniformly bounded inverse everywhere in the open half-plane $\Re \sigma < 0$.

**Proof.** According to the arguments advanced before the derivation of (3), $L$ has a bounded inverse if and only if the operator $S$ enjoys the same property. It follows from elementary Hilbert space theory that $L$ has a bounded inverse if and only if the adjoint operator $S^*$ defined by (5) has a
bounded inverse on $H^2(\mathcal{K})$. Moreover, since the linear fractional transformation $z = (2\sigma + 1)(2\sigma - 1)^{-1}$ maps the half-plane $\text{Re} \sigma < 0$ onto the disc $|z| < 1$, we see from Rabindranathan's lemma that $S^*$ has a bounded inverse if and only if the operator-valued polynomial

$$R \left( \frac{2\sigma + 1}{2\sigma - 1} \right) = \sum A_k^* \sigma^k = P(\sigma)$$

has an analytic inverse which is uniformly bounded everywhere in the half-plane $\text{Re} \sigma < 0$. But the inverse of an operator-valued polynomial is clearly analytic, and this completes the proof.

A special case of our theorem deserves attention because of its utility in the study of matrix differential equations.

**Corollary.** When the dimension of $\mathcal{K}$ is finite, $L$ has a bounded inverse if and only if the determinant of $P(\sigma)$ has no zeros in the closed left half-plane.

One final remark should be made at this point: It seems quite probable that our techniques can be extended to cope with differential equations having *unbounded* coefficients. Some results in this direction are now being prepared for later publication.

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**References**