ON THE ARENS PRODUCT AND COMMUTATIVE BANACH ALGEBRAS

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Abstract. The purpose of this note is to generalize two recent results by the author for commutative Banach algebras. Let $A$ be a commutative Banach algebra with carrier space $X_A$ and $\pi$ the canonical embedding of $A$ into its second conjugate space $A^{**}$ (with the Arens product). We show that if $A$ is a semisimple annihilator algebra, then $\pi(A)$ is a two-sided ideal of $A^{**}$. We also obtain that if $A$ is a dense two-sided ideal of $C_0(X_A)$, then $\pi(A)$ is a two-sided ideal of $A^{**}$ if and only if $A$ is a modular annihilator algebra.

1. Notation and preliminaries. Notation and definition not explicitly given are taken from Rickart's book [5].

For any subset $E$ of a Banach algebra $A$, let $L_A(E)$ and $R_A(E)$ denote the left and right annihilators of $E$ in $A$, respectively. Then $A$ is called a modular annihilator algebra if, for every maximal modular left ideal $I$ and for every maximal modular right ideal $J$, we have $R_A(I)=(0)$ if and only if $I=A$ and $L_A(J)=(0)$ if and only if $J=A$ (see [2, p. 568, Definition]).

Let $A$ be a Banach algebra, $A^*$ and $A^{**}$ the conjugate and second conjugate spaces of $A$, respectively. The Arens product on $A^{**}$ is defined in stages according to the following rules (see [1]). Let $x, y \in A, f \in A^*, F, G \in A^{**}$.

(a) Define $f \circ x$ by $(f \circ x)(y)=f(xy)$. Then $f \circ x \in A^*$.

(b) Define $G \circ f$ by $(G \circ f)(x)=G(f \circ x)$. Then $G \circ f \in A^*$.

(c) Define $F \circ G$ by $(F \circ G)(f)=F(G \circ f)$. Then $F \circ G \in A^{**}$.

$A^{**}$ with the Arens product $\circ$ is denoted by $(A^{**}, \circ)$. Let $\pi$ be the canonical embedding of $A$ into $A^{**}$. Then $\pi(A)$ is a subalgebra of $(A^{**}, \circ)$.

Let $A$ be a Banach algebra. For each element $x \in A$, let $Sp_A(x)$ denote the spectrum of $x$ in $A$. If $A$ is commutative, $X_A$ will denote the carrier space of $A$ and $C_0(X_A)$ the algebra of all complex-valued continuous functions on $X_A$, which vanish at infinity; $C_0(X_A)$ is a commutative $B^*$-algebra.

In this paper, all algebras and linear spaces under consideration are over the complex field $C$.
2. The results. Our first result is a generalization of [7, p. 82, Theorem 3.3] for commutative Banach algebras.

**Theorem 2.1.** Let $A$ be a semisimple commutative annihilator Banach algebra. Then $A$ is a two-sided ideal of $(A^{**}, \circ)$.

**Proof.** Let $X_A$ be the carrier space of $A$ and let $B = C_0(X_A)$. Since $A$ is an annihilator algebra, it is well known that $X_A$ is discrete and therefore $B$ is a dual $B^*$-algebra by [6, p. 532, Theorem 4.2]. Let $|\cdot|$ be the norm on $B$. Then the given norm $||\cdot||$ majorizes $|\cdot|$ on $A$. Considering $A$ as a subalgebra of $B$, we show that $A$ is dense in $B$. Let $x \not= 0$ be a hermitian element in $B$. Then it is known that $\text{Sp}_B(x)$ has no nonzero limit points and so it is a countable set (see [8, p. 826, Theorem 3.1]). Therefore it follows from [5, p. 111, Theorem (3.1.6)] that $\{\alpha(x) : \alpha \in X_A\}$ is countable. Denote those $\alpha \in X_A$ for which $\alpha(x) \not= 0$ by $\alpha_1, \alpha_2, \ldots$. Then $\alpha_n(x) \to 0$ as $n \to \infty$. For each $\alpha_n$, by Silov’s theorem [5, p. 168, Theorem (3.6.3)], there exists a nonzero idempotent $e_n$ in $A$ such that $e_n(\alpha_n) = 1$ and $e_n(\alpha) = 0$ for all $\alpha \not= \alpha_n$.

Let $\varepsilon > 0$ be given. Then there exists a positive integer $N$ such that $|\alpha_n(x)| < \varepsilon$ for all $n \geq N$. Let $y = \sum_{n=1}^{N} \alpha_n(x)e_n$. Then $y \in A$ and it is easy to see that

$$|x - y| = \sup\{|\alpha(x) - \alpha(y)| : \alpha \in X_A\} < \varepsilon.$$ 

Therefore it follows now that $A$ is dense in $B$. Since $A$ is a dual $B^*$-algebra, by [7, p. 82, Theorem 3.3], $B$ is a two-sided ideal of $B^{**}$ (with the Arens product). Therefore, by [7, p. 82, Lemma 3.2], $A$ is a two-sided ideal of $(A^{**}, \circ)$ and the proof is complete.

The following result is a generalization of [8, p. 830, Theorem 5.2] for commutative Banach algebras.

**Theorem 2.2.** Let $A$ be a commutative Banach algebra such that $A$ is a dense two-sided ideal of $C_0(X_A)$. Then $A$ is a two-sided ideal of $(A^{**}, \circ)$ if and only if $A$ is a modular annihilator algebra.

**Proof.** Let $B = C_0(X_A)$ and let $|\cdot|$ be the norm on $B$. By [3, p. 3, Theorem 2.3], there exists a constant $k$ such that $|ab| \leq k|a| |b|$ and $|ab| \leq k|a| |b|$ for all $a, b \in A$. It is easy to see that $A$ is a semisimple algebra. Suppose $A$ is a two-sided ideal of $(A^{**}, \circ)$. Then by the proof of [8, p. 829, Lemma 5.1], we can show that $X_A$ is discrete and therefore $A$ is a modular annihilator algebra (see [2, p. 578, Example 8.4]). Conversely suppose $A$ is a modular annihilator algebra. Then, by [2, p. 569, Theorem 4.2 (6)], $X_A$ is discrete in the hull-kernel topology and therefore $X_A$ is discrete in the finer Gelfand topology. Hence $B$ is a dual $B^*$-algebra. Now by the argument in [8, p. 830, Theorem 5.2], we can show that $A$ is a two-sided ideal of $(A^{**}, \circ)$, and this completes the proof.
**Corollary 2.3.** Let $A$ be as in Theorem 2.2. If $A$ is reflexive and has an approximate identity, then $A$ is finite dimensional.

**Proof.** By [4, p. 855, Lemma 3.8], $A$ has an identity element. Also it follows from Theorem 3.2 that $A$ is a modular annihilator algebra and therefore it is finite dimensional by [2, p. 573, Proposition 6.3].

Let $G$ be a compact abelian group with the Haar measure and let $A = L_2(G)$. Then it is well known that $A$ is reflexive and $A$ is a dense two-sided ideal of $C_0(X_A)$. Also if $A$ is infinite dimensional, $A$ has no approximate identity.

**References**


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