A CHARACTERIZATION OF DUAL $B^*$-ALGEBRAS

EDITH A. McCHAREN

ABSTRACT. Let $A$ be a $B^*$-algebra. The second conjugate space of $A$, denoted by $A^{**}$, is a $B^*$-algebra under the Arens multiplication. A new proof is given that $A$ is a dual algebra if and only if the natural image of $A$ in $A^{**}$ is an ideal in $A^{**}$.

In this note we present an alternate proof to the following result of B. J. Tomiuk and Pak-Ken Wong [1].

THEOREM. Let $A$ be a (complex) $B^*$-algebra, $A^{**}$ be its second conjugate space and $\pi$ the canonical imbedding of $A$ into $A^{**}$. Then $A$ is a dual algebra if and only if $\pi(A)$ is an ideal in $A^{**}$ with respect to the Arens multiplication.

PROOF. For each $a \in A$ let $L_a$ and $R_a$ denote the operators on $A$ defined by $L_a(x) = ax$ and $R_a(x) = xa$. If $A^*$ denotes the conjugate space of $A$, let $L_a^*: A^* \to A^*$ denote the transpose of $L_a$ defined by $L_a^*(f)(x) = f(L_a(x)) (f \in A^*, x \in A)$. The second transpose of $L_a$ is then the mapping $L_a^{**}: A^{**} \to A^{**}$ defined by $L_a^{**}(F)(f) = F(L_a^*(f)) (F \in A^{**}, f \in A^*)$. Let $R_a^*$ denote the second transpose of $R_a$ defined similarly.

Then $A$ is a dual algebra if and only if $L_a$ and $R_a$ are weakly compact operators for each $a \in A$ [2, p. 99]. By the generalized Gantmacher theorem [3, pp. 624-625], $L_a$ and $R_a$ ($a \in A$) are weakly compact if and only if $L_a^{**}(A^{**}) \cup R_a^{**}(A^{**}) \subseteq \pi(A)$. Let $\ast$ denote the Arens multiplication in $A^{**}$. It follows from the definition of this multiplication (see [1, p. 530]) that $L_a^*(F) = \pi(a) \ast F$ and $R_a^*(F) = F \ast \pi(a)$ for all $F \in A^{**}$, and therefore $L_a^{**}(A^{**}) \cup R_a^{**}(A^{**}) = \pi(a) \ast A^{**} \cup A^{**} \ast \pi(a)$, which completes the proof.

REFERENCES


DEPARTMENT OF MATHEMATICS, NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115

Received by the editors September 28, 1970 and, in revised form, May 13, 1971.