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A THEOREM ON INJECTIVITY OF THE CUP PRODUCT

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ABSTRACT. We prove that if a space X has abelian or sufficiently abelian fundamental group, then the cup product $H^1(X)\wedge H^1(X)\to H^2(X)$ is injective, giving an inequality between the associated Betti numbers. This generalises to a theorem of injectivity of the k-fold cup product on $H^n(X)$, given that the kth order Whitehead product on $\pi_n(X)$ is trivial or torsion.

Let X be a topological space, and let $H_n(X)$ denote its nth singular integral homology group. Let $\mathscr{A}$ denote the class of groups π such that $p:\pi\to\pi/[\pi,\pi]$ splits rationally, i.e. there exists a homomorphism $q:\pi/[\pi,\pi]\to\pi$ such that $pq\otimes 1:\pi/[\pi,\pi]\otimes\boldsymbol{Q}\to\pi/[\pi,\pi]\otimes\boldsymbol{Q}$ is an isomorphism.

Set $\odot^k H^n(X)$ equal to, if n is odd, the kth exterior power $\bigwedge^k H^n(X)$, and if n is even, the kth symmetric power of $H^n(X)$, then:

THEOREM 1. *The cup product pairing*: $\odot^k H^n(X)\to H^{kn}(X)$ *is injective* $(n\geqq 1, k\geqq 2)$ *if*:

(a) *the Hurewicz homomorphism*: $\pi_n(X)\to H_n(X)$ *is epimorphic*,

(b) $H_{n-1}(X)$ *is a free* $\boldsymbol{Z}$*-module*,

(c) (*for* $(n,k)\neq(1,2)$) *the kth order Whitehead product*: $\pi_n(X)\times\cdots\times\pi_n(X)\to\pi_{kn-1}(X)$ *is trivial or torsion* (*i.e. each Whitehead product contains zero or a torsion element*),

(c′) (*for* $(n,k)=(1,2)$) $\pi_1(X)\in\mathscr{A}$.

Note. The class $\mathscr{A}$ includes abelian groups, periodic groups, and also such nearly abelian groups as the group generated by a, b under the single relation: $a^n b=ba^n$, this being π_1 of the space $(S_1^1\vee S_2^1)\cup_{[ni_1,i_2]}e^2$.

COROLLARY 1. *Under the hypotheses of Theorem* 1, *the Betti numbers* β_r *satisfy the following inequality*:

$$\binom{\beta_n+k-1}{k}\leqq\beta_{kn},\quad n\text{ even},\qquad \binom{\beta_n}{k}\leqq\beta_{kn},\quad n\text{ odd}.$$

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COROLLARY 2. *If* $\pi_1(X) \in \mathscr{A}$, *the cup product*: $H^1(X) \wedge H^1(X) \to H^2(X)$ *is injective, and thus* $\frac{1}{2}\beta_1(\beta_1 - 1) \leqq \beta_2$.

Corollary 2 was proved by K.-T. Chen [1] for a differentiable manifold with abelian fundamental group, using real or complex deRham cohomology. A proof valid for CW complexes with finitely generated homotopy groups and cohomology coefficients in any field is given by Massey in a footnote to [1]. The inequality on the Betti numbers was first noted by Hopf [2] for a simplicial complex with abelian fundamental group.

COROLLARY 3. *If* X *is a compact connected* 3-*manifold with* $\pi_1(X) \in \mathscr{A}$, *then* $\beta_1 \leqq 3$.

K. Reidemeister proved this for $\pi_1(X)$ abelian in [3].

COROLLARY 4. *If* X *is a compact connected orientable* 4-*manifold with* $\pi_1(X) \in \mathscr{A}$, *then the Euler number of* $X \geqq -1$.

PROOF OF THEOREM 1. We use the following Lemma whose proof is omitted:

LEMMA. *Let* E *be a* $\boldsymbol{Z}$-*module, and let* $x^1, \cdots, x^m$ *be linearly independent elements of* $\mathrm{Hom}(E, \boldsymbol{Z})$. *Then there exists a positive integer* K, *linearly independent elements* $y^1, \cdots, y^m \in \mathrm{Hom}(E, \boldsymbol{Z})$, *and* $e_1, \cdots, e_m \in E$, *such that* $Kx^1, \cdots, Kx^m$ *are linearly equivalent*[2] *to* $y^1, \cdots, y^m$, *and* $\langle y^i, e_j \rangle = K\delta_{ij}$.

We proceed to the proof of Theorem 1.

By hypothesis (b), $H^n(X) \cong \mathrm{Hom}(H_n(X), \boldsymbol{Z})$. Suppose $x^1, \cdots, x^m$ are linearly independent elements of $H^n(X)$. By the Lemma there exists integer K, $y^1, \cdots, y^m \in H^n(X)$, linearly equivalent to $Kx^1, \cdots, Kx^m$, and $e_1, \cdots, e_m \in H_n(X)$ such that $\langle y^i, e_j \rangle = K\delta_{ij}$; to prove injectivity of the cup product, it suffices to prove that the elements

$$\text{(i)} \qquad y^{i_1} \cup \cdots \cup y^{i_k}, \qquad 1 \leqq i_1 < \cdots < i_k \leqq m$$

(with $<$ replaced by $\leqq$ when n is even) are linearly independent in $H^{kn}(X)$.

We choose an integer P and elements $a_1, \cdots, a_m \in \pi_n(X)$ with homology classes $Pe_1, \cdots, Pe_m$ such that each Whitehead product $[a_{i_1}, \cdots, a_{i_k}]$ contains zero, as follows:

If $(n, k) = (1, 2)$ choose $a'_1, \cdots, a'_m \in \pi/[\pi, \pi] \otimes \boldsymbol{Q}$ such that $pq \otimes 1(a'_i)$ has homology class $e_i \otimes 1$, choose P so that each Pa'_i actually lies in $\pi/[\pi, \pi]$ and set $a_i = q(Pa'_i)$.

[2] I.e. each set is linearly dependent on the other.

If $(n, k) \neq (1, 2)$ choose $a'_1, \cdots, a'_m \in \pi_n(X)$ with homology classes $e_1, \cdots, e_m$. By hypothesis, each Whitehead product $[a'_{i_1}, \cdots, a'_{i_k}]$ contains a torsion element. Let P be chosen so that $0 \in P[a'_{i_1}, \cdots, a'_{i_k}]$ for all $i_1, \cdots, i_k$. Set $a_i = Pa'_i$. It follows from [4] that $0 \in [a_{i_1}, \cdots, a_{i_k}]$.

Form the wedge of maps representing $a_{i_1}, \cdots, a_{i_k}: S^n \vee \cdots \vee S^n \to X$; then according to Porter [4, Theorem 2.4], hypothesis (c) allows us to extend to a map $k: S^n \times \cdots \times S^n \to X$.

By naturality $k^*(y^{j_1} \cup \cdots \cup y^{j_k}) = k^* y^{j_1} \cup \cdots \cup k^* y^{j_k}$. Evaluating the $k^* y^{j_i}$ on the generators of $H_n(S^n \times \cdots \times S^n)$ we have:

$$\begin{aligned} k^*(y^{j_1} \cup \cdots \cup y^{j_k}) &= 0 && (j_1, \cdots, j_k) \neq (i_1, \cdots, i_k), \\ &= (KP)^k M && (j_1, \cdots, j_k) = (i_1, \cdots, i_k), \end{aligned}$$

with $M = 1$ if n is odd; or $M = |N_1|! \cdots |N_t|!$, if n is even, where $N_1, \cdots, N_t$ is a partition of the indices $1, \cdots, k$ under the equivalence relation $r \sim s$ if and only if $j_r = j_s$. This shows that the coefficient of each $x^{i_1} \cup \cdots \cup x^{i_k}$ in any relation of linear dependence of the elements (i) must be zero, hence injectivity of the cup product is established.

Notes. (1) If R is a ring which is torsion free as a $\boldsymbol{Z}$-module, then under the hypotheses of Theorem 1, the universal coefficient formula shows that the cup product: $\odot^k H^n(X; R) \to H^{kn}(X; R)$ is injective. When $R = \boldsymbol{Q}$, a slight modification of the proof of Theorem 1 using $H^n(X, \boldsymbol{Q}) = \mathrm{Hom}(H_n(X, \boldsymbol{Z}), \boldsymbol{Q})$ shows that hypothesis (b) is unnecessary for injectivity: $\odot^k H^n(X, \boldsymbol{Q}) \to H^{kn}(X, \boldsymbol{Q})$. Note this shows Corollary 1 is true without hypothesis (b).

(2) The theorem is, in spirit, the contrapositive of Theorem 3.3 of [4], and uses a similar calculation in its proof. The essential idea is that a Whitehead product which is trivial or torsion forces a cup product to be nonzero. As an example of how we can create a cup product by killing a Whitehead product, let X be the space formed by attaching a $2n$-cell to S^n (n even) via the Whitehead square of the generator of $\pi_n S^n$. If x is the generator of $H^n(X)$, a simple calculation similar to that used in Theorem 1 shows $x \cup x =$ twice the generator of $H^{2n}(X)$. The opposite process of creating a Whitehead product by killing a cup product is expounded by Porter in [5, Theorem 3.15].

(3) To obtain a version of Theorem 1 for coefficients in $\boldsymbol{Z}_p$, p prime, we must replace hypothesis (a) by the assumption that the composite: $\pi_n(X) \to H_n(X) \to H_n(X; \boldsymbol{Z}_p)$ is surjective, remove hypothesis (b), and replace hypothesis (c) by:

(if $(n, k) = (1, 2)$) $p: \pi \to \pi/[\pi, \pi]$ splits mod p, i.e. there exists $q: \pi/[\pi, \pi] \to \pi$ such that $pq \otimes 1: \pi/[\pi, \pi] \otimes \boldsymbol{Z}_p \to \pi/[\pi, \pi] \otimes \boldsymbol{Z}_p$ is an isomorphism,

(if $(n, k) \neq (1, 2)$) each Whitehead product $[a_1, \cdots, a_k]$, where $a_i \in \pi_n(X)$, contains a torsion element of order prime to p.

We conclude that the cup product: $\odot^k H^n(X; Z_p) \to H^{kn}(X; Z_p)$ is injective provided either n is odd or n is even and $k < p$. This last restriction ensures that the M in the proof of Theorem 1 is nonzero mod p, and its necessity is shown by the example in note (2) for $k=2$, $p=2$.

(4) A similar theorem asserting that the cup product is injective on a suitable factor space of $H^{n_1}(X) \otimes \cdots \otimes H^{n_k}(X)$ can be proved under the hypotheses (a), (b) of Theorem 1 for each $n = n_i$, and the hypothesis that the kth order Whitehead product:

$$\pi_{n_1}(X) \times \cdots \times \pi_{n_k}(X) \to \pi_{n_1+\cdots+n_k-1}(X)$$

is trivial or torsion.

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