A THEOREM ON INJECTIVITY OF THE CUP PRODUCT

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ABSTRACT. We prove that if a space $X$ has abelian or sufficiently abelian fundamental group, then the cup product $H^1(X) \wedge H^1(X) \rightarrow H^2(X)$ is injective, giving an inequality between the associated Betti numbers. This generalizes to a theorem of injectivity of the $k$-fold cup product on $H^n(X)$, given that the $k$th order Whitehead product on $\pi_n(X)$, given that the $k$th order Whitehead product on $\pi_n(X)$ is trivial or torsion.

Let $X$ be a topological space, and let $H_n(X)$ denote its $n$th singular integral homology group. Let $\mathcal{A}$ denote the class of groups $\pi$ such that $p: \pi \rightarrow \pi/[\pi, \pi]$ splits rationally, i.e. there exists a homomorphism $q: \pi/[\pi, \pi] \rightarrow \pi$ such that $pq \otimes 1: \pi/[\pi, \pi] \otimes \mathbb{Q} \rightarrow \pi/[\pi, \pi] \otimes \mathbb{Q}$ is an isomorphism.

Set $\bigwedge^k H^n(X)$ equal to, if $n$ is odd, the $k$th exterior power $\bigwedge^k H^n(X)$, and if $n$ is even, the $k$th symmetric power of $H^n(X)$, then:

THEOREM 1. The cup product pairing: $\bigwedge^k H^n(X) \rightarrow H^{kn}(X)$ is injective ($n \geq 1, k \geq 2$) if:

(a) the Hurewicz homomorphism: $\pi_n(X) \rightarrow H_n(X)$ is epimorphic,

(b) $H_{n-1}(X)$ is a free $\mathbb{Z}$-module,

(c) (for $(n, k) \neq (1, 2)$) the $k$th order Whitehead product: $\pi_n(X) \times \cdots \times \pi_n(X) \rightarrow \pi_{kn-1}(X)$ is trivial or torsion (i.e. each Whitehead product contains zero or a torsion element),

(c') (for $(n, k) = (1, 2)$) $\pi_1(X) \in \mathcal{A}$.

Note. The class $\mathcal{A}$ includes abelian groups, periodic groups, and also such nearly abelian groups as the group generated by $a, b$ under the single relation: $a^n b = b a^n$, this being $\pi_1$ of the space $(S^1 \vee S^1) \cup \{n_1, n_2\} e^2$.

COROLLARY 1. Under the hypotheses of Theorem 1, the Betti numbers $\beta$, satisfy the following inequality:

$$\binom{\beta_n + k - 1}{k} \leq \beta_{kn}, \quad n \text{ even,} \quad \binom{\beta_n}{k} \leq \beta_{kn}, \quad n \text{ odd.}$$
Corollary 2. If \( \pi_1(X) \in \mathcal{A} \), the cup product: \( H^1(X) \wedge H^1(X) \to H^2(X) \)

is injective, and thus \( \frac{1}{2} \beta_1(\beta_1 - 1) \leq \beta_2 \).

Corollary 2 was proved by K.-T. Chen [1] for a differentiable manifold with abelian fundamental group, using real or complex deRham cohomology. A proof valid for CW complexes with finitely generated homotopy groups and cohomology coefficients in any field is given by Massey in a footnote to [1]. The inequality on the Betti numbers was first noted by Hopf [2] for a simplicial complex with abelian fundamental group.

Corollary 3. If \( X \) is a compact connected 3-manifold with \( \pi_1(X) \in \mathcal{A} \), then \( \beta_1 \leq 3 \).

K. Reidemeister proved this for \( \pi_1(X) \) abelian in [3].

Corollary 4. If \( X \) is a compact connected orientable 4-manifold with \( \pi_1(X) \in \mathcal{A} \), then the Euler number of \( X \) is \(-1 \).

Proof of Theorem 1. We use the following Lemma whose proof is omitted:

**Lemma.** Let \( E \) be a \( \mathbb{Z} \)-module, and let \( x_1, \ldots, x_m \) be linearly independent elements of \( \text{Hom}(E, \mathbb{Z}) \). Then there exists a positive integer \( K \), linearly independent elements \( y_1, \ldots, y_m \in \text{Hom}(E, \mathbb{Z}) \), and \( e_1, \ldots, e_m \in E \), such that \( Kx_1, \ldots, Kx_m \) are linearly equivalent to \( y_1, \ldots, y_m \), and \( \langle y_i, e_j \rangle = K \delta_{ij} \).

We proceed to the proof of Theorem 1.

By hypothesis (b), \( H^n(X) \cong \text{Hom}(H_n(X), \mathbb{Z}) \). Suppose \( x_1, \ldots, x_m \) are linearly independent elements of \( H^n(X) \). By the Lemma there exists integer \( K \), \( y_1, \ldots, y_m \in H^n(X) \), linearly equivalent to \( Kx_1, \ldots, Kx_m \), and \( e_1, \ldots, e_m \in H_n(X) \) such that \( \langle y_i, e_j \rangle = K \delta_{ij} \); to prove injectivity of the cup product, it suffices to prove that the elements

(i) \( y^{i_1} \cup \cdots \cup y^{i_k} \), \( 1 \leq i_1 < \cdots < i_k \leq m \)

(with \( < \) replaced by \( \leq \) when \( n \) is even) are linearly independent in \( H^{kn}(X) \).

We choose an integer \( P \) and elements \( a_1, \ldots, a_m \in \pi_n(X) \) with homology classes \( Pa_1, \ldots, Pa_m \) such that each Whitehead product \( [a_{i_1}, \ldots, a_{i_k}] \) contains zero, as follows:

If \( (n, k) = (1, 2) \) choose \( a_1', \ldots, a_m' \in \pi[\pi, \pi] \otimes Q \) such that \( pq \otimes 1(a_i') \) has homology class \( e \otimes 1 \), choose \( P \) so that each \( Pa_i' \) actually lies in \( \pi[\pi, \pi] \) and set \( a_i = q(Pa_i') \).

\(^2\) I.e. each set is linearly dependent on the other.
If \((n, k) \neq (1, 2)\) choose \(a'_1, \ldots, a'_m \in \pi_n(X)\) with homology classes \(e_1, \ldots, e_m\). By hypothesis, each Whitehead product \([a'_{i_1}, \ldots, a'_{i_k}]\) contains a torsion element. Let \(P\) be chosen so that \(0 \in P[a'_{i_1}, \ldots, a'_{i_k}]\) for all \(i_1, \ldots, i_k\). Set \(a_i = Pa'_i\). It follows from [4] that \(0 \in [a_1, \ldots, a_k]\).

Form the wedge of maps representing \(a_1, \ldots, a_k : S^n \vee \cdots \vee S^n \to X\); then according to Porter [4, Theorem 2.4], hypothesis (c) allows us to extend to a map \(k : S^n \times \cdots \times S^n \to X\).

By naturality \(k^*(y^{j_1} \cup \cdots \cup y^{j_k}) = k^*y^{j_1} \cup \cdots \cup k^*y^{j_k}\). Evaluating the \(k^*y^{j_i}\) on the generators of \(H_n(S^n \times \cdots \times S^n)\) we have:

\[
k^*(y^{j_1} \cup \cdots \cup y^{j_k}) = 0 \quad (j_1, \ldots, j_k) \neq (i_1, \ldots, i_k),
\]

\[
= (KP)k^*M \quad (j_1, \ldots, j_k) = (i_1, \ldots, i_k),
\]

with \(M = 1\) if \(n\) is odd; or \(M = |N_1| \cdots |N_k|\), if \(n\) is even, where \(N_1, \ldots, N_i\) is a partition of the indices \(1, \ldots, k\) under the equivalence relation \(r \sim s\) if and only if \(j_r = j_s\). This shows that the coefficient of each \(x^{i_1} \cup \cdots \cup x^{i_k}\) in any relation of linear dependence of the elements \((i)\) must be zero, hence injectivity of the cup product is established.

Notes. (1) If \(R\) is a ring which is torsion free as a \(Z\)-module, then under the hypotheses of Theorem 1, the universal coefficient formula shows that the cup product: \(\bigodot^k H^n(X; R) \to H^{kn}(X; R)\) is injective. When \(R = Q\), a slight modification of the proof of Theorem 1 using \(H^n(X, Q) = Hom(H_n(X, Z), Q)\) shows that hypothesis (b) is unnecessary for injectivity: \(\bigodot^k H^n(X, Q) \to H^{kn}(X, Q)\). Note this shows Corollary 1 is true without hypothesis (b).

(2) The theorem is, in spirit, the contrapositive of Theorem 3.3 of [4], and uses a similar calculation in its proof. The essential idea is that a Whitehead product which is trivial or torsion forces a cup product to be nonzero. As an example of how we can create a cup product by killing a Whitehead product, let \(X\) be the space formed by attaching a \(2n\)-cell to \(S^n\) \((n\) even) via the Whitehead square of the generator of \(\pi_n S^n\). If \(x\) is the generator of \(H^n(X)\), a simple calculation similar to that used in Theorem 1 shows \(x \cup x =\) twice the generator of \(H^{2n}(X)\). The opposite process of creating a Whitehead product by killing a cup product is expounded by Porter in [5, Theorem 3.15].

(3) To obtain a version of Theorem 1 for coefficients in \(Z_p\), \(p\) prime, we must replace hypothesis (a) by the assumption that the composite: \(\pi_n(X) \to H_n(X) \to H_n(X; Z_p)\) is surjective, remove hypothesis (b), and replace hypothesis (c) by:

- (if \((n, k) = (1, 2)\)) \(p : \pi_n \to \pi_n / [\pi, \pi]\) splits mod \(p\), i.e. there exists \(q : \pi_n / [\pi, \pi] \to \pi\) such that \(pq \otimes 1 : \pi_n / [\pi, \pi] \otimes Z_p \to \pi / [\pi, \pi] \otimes Z_p\) is an isomorphism,

- (if \((n, k) \neq (1, 2)\)) each Whitehead product \([a_1, \ldots, a_k]\), where \(a_i \in \pi_n(X)\), contains a torsion element of order prime to \(p\).
We conclude that the cup product: $\bigotimes^k H^n(X; \mathbb{Z}_p) \to H^{kn}(X; \mathbb{Z}_p)$ is injective provided either $n$ is odd or $n$ is even and $k < p$. This last restriction ensures that the $M$ in the proof of Theorem 1 is nonzero mod $p$, and its necessity is shown by the example in note (2) for $k=2$, $p=2$.

(4) A similar theorem asserting that the cup product is injective on a suitable factor space of $H^{n_1}(X) \otimes \cdots \otimes H^{n_k}(X)$ can be proved under the hypotheses (a), (b) of Theorem 1 for each $n=n_i$, and the hypothesis that the $k$th order Whitehead product:

$$\pi_{n_1}(X) \times \cdots \times \pi_{n_k}(X) \to \pi_{n_1+\cdots+n_k-1}(X)$$

is trivial or torsion.

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References


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