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ADJOINTS OF MULTIPOINT-INTEGRAL BOUNDARY VALUE PROBLEMS

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ABSTRACT. The dual system to $Ly=y'+Py$,

$$\sum_{i=0}^{\infty} A_i y(t_i) + \int_0^1 K(t)y(t)\,dt = 0$$

is found when the setting is $L_n^p(0, 1)$, $1<p<\infty$.

Introduction. Let X denote the Banach space $L_n^p(0, 1)$, consisting of n-dimensional vectors under the norm

$$\|x\| = \left[\int_0^1 \left[\sum_{i=1}^{n} |x_i(t)|^2\right]^{p/2} dt\right]^{1/p},$$

$1<p<\infty$. In X and X^* $(=L_n^q(0, 1), 1/p+1/q=1)$ let us define the following subspaces:

(a) Let D' denote those vectors in X which are absolutely continuous.

(b) Let D_0^+ denote those vectors in X^* which are absolutely continuous and vanish at $t=0$, $t=1$.

(c) Let A_i, $i=0, \ldots,$ be $m\times n$ constant matrices satisfying $\sum_{i=0}^{\infty} \|A_i\|<\infty$ ($\|\cdot\|$ here denotes a convenient norm) and $\bigcap_{i=0}^{\infty} \ker A_i^*=0$. Let $\{t_i\}_{i=0}^{\infty}$ be a collection of points in $[0, 1]$ $(t_0=0, t_1=1)$, and let $K(t)$ be an $m\times n$ matrix valued function whose rows are in X^*. For all y in D' we define the (discontinuous) boundary functional V by

$$Vy = \sum_{i=0}^{\infty} A_i y(t_i) + \int_0^1 K(t)y(t)\,dt,$$

and denote by D the kernel of this functional. $Vy=0$ for all $y\in D$.

(d) Let D^+ denote those vectors z in X^* for which there exists an $m\times 1$ matrix valued functional $\phi(z)$ such that

(1) $z(t)+\sum_{i=0}^{\infty} A_i^*\phi(z)\lambda(t_i, 1]$ is absolutely continuous on $[0, 1]$. ($\lambda(t_i, 1]$ is the characteristic function of $(t_i, 1]$.)

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(2) If $P(t)$ is a continuous $n \times n$ matrix valued function, then the expression $l^+z = -z' + P^*z + K^*\phi(z)$ exists a.e. and is in X^*.

We note that ϕ is uniquely defined, for if two such ϕ, ϕ_1 and ϕ_2, exist, then by subtraction $\sum_{i=0}^{\infty} A_i^*(\phi_1 - \phi_2)\lambda(t_i, 1]$ would be absolutely continuous. This is impossible. Further note that the range of ϕ is all of R^m, that $D_0^+ \subset D^+$, and that $\phi(z)=0$ if and only if $z \in D_0^+$. Hence D^+ is nonempty.

Finally, letting $ly = y' + Py$, l^+z as given above, we define the following operators:

(a) L' is given by $L'y = ly$ for all y in D'.

(b) L_0^+ is given by $L_0^+ z = l^+z$ for all z in D_0^+.

(c) L is given by $Ly = ly$ for all y in D.

(d) L^+ is given by $L^+z = l^+z$ a.e. for all z in D^+.

Our principal result is that L and L^+ are dual operators. Brown **[1]** has previously shown that D is dense in X, so L^* exists. Further Bryan **[2]** used L^+ as the basis of a definition of L^*. The result is well known under the classical endpoint conditions. However, because of the possible density of the boundary points $\{t_i\}$ in the interval [0, 1], the techniques now used are different from those used previously.

The result also generalizes similar results recently obtained by Green and Krall **[4]** and Krall **[5]**.

The closure of L^+.

THEOREM 1. *The operator L^+, defined by $L^+z = -z' + P^*z + K^*\phi(z)$, for all $z \in D^+$, is closed.*

PROOF. We recall that the operator L^+ is closed if when $\lim_{k\to\infty} z_k = z$ ($z_k \in D^+$), and $\lim_{k\to\infty} L^+z_k = y$, then $z \in D^+$, and $L^+z = y$.

Let $\varepsilon(s) = \sum_{i=0}^{\infty} A_i^*\lambda(t_i, 1](s)$. Then

$$\begin{aligned} z_k(s) - z_l(s) &= \int_0^s (z_k' - z_l')\, d\xi + \varepsilon(s)\phi(z_l - z_k) \\ &= \int_0^s L^+(z_l - z_k)\, d\xi + \int_0^s P^*(z_k - z_l)\, d\xi \\ &\quad + \left[\int_0^s K^*\, d\xi - \varepsilon(s)\right]\phi(z_k - z_l). \end{aligned}$$

Since

$$\left\| \int_0^s L^+(z_k - z_l)\, d\xi \right\|_{X^*} \leqq \|L^+(z_l - z_k)\|_{X^*}$$

and

$$\left\| \int_0^s P^*(z_k - z_l)\, d\xi \right\|_{X^*} \leqq \|P^*\|\ \|z_k - z_l\|_{X^*}$$

by Hölder's inequality, we have

$$\left\| \left[\int_0^s K^* \, d\xi - \varepsilon(s) \right] \phi(z_k - z_l) \right\|_{X^*} \leqq \|z_k - z_l\|_{X^*} + \|P^*\| \, \|z_k - z_l\|_{X^*} + \|L^+(z_l - z_k)\|_{X^*}.$$

By assumption, each of the terms on the right approaches 0 as k, l approach ∞. Therefore the functions

$$F_k(s) = \left[\int_0^s K^* \, d\xi - \varepsilon(s) \right] \phi(z_k)$$

converge in measure. Hence there is a subsequence $F_{k_j}(s)$ which converges almost everywhere. Letting s approach t_i from above and below implies $F_{k_j}(t_i^+) - F_{k_j}(t_i^-) = A_i^* \phi_{k_j}$ converges. Therefore $\phi_{k_j} - \phi_{k_l}$[1] approaches the kernel of A_i^* as k_j, k_l approach ∞. Since this holds for all i, and $\bigcap_{i=0}^{\infty} \ker A_i^* = 0$, we conclude that $\phi_{k_j} - \phi_{k_l}$ approaches 0. In other words ϕ_{k_j} converges.

Since

$$z_{k_j}(s) = -\int_0^s L^+ z_{k_j} \, d\xi + \int_0^s P^* z_{k_j} \, d\xi + \left[\int_0^s K^* \, d\xi - \varepsilon(s) \right] \phi(z_{k_j})$$

and $\phi(z_{k_j})$ converges, we may take limits to find

$$z(s) = -\int_0^s y \, d\xi + \int_0^s P^* z \, d\xi + \left[\int_0^s K^* \, d\xi - \varepsilon(s) \right] \phi,$$

or

$$z(s) + \varepsilon(s)\phi = -\int_0^s y \, d\xi + \int_0^s P^* z \, d\xi + \int_0^s K^* \, d\xi \, \phi.$$

Since the right is absolutely continuous, z satisfies the first requirement of elements in D^+. Differentiating, we find $z' = -y + P^* z + K^* \phi$, or $y = l^+ z$ a.e., the second condition.

The duals of L and L^+. Let us denote the dual of an operator A by A^*.

LEMMA. $(L_0^+)^* = L'$.

PROOF. This is well known. See Goldberg [**3**, p. 51].

THEOREM 2. $(L^+)^* = L$.

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[1] The matrix A_i^* defines an isomorphism f_i from the factor space $R^m/\ker A_i^*$ onto the range of A_i^*. Thus $A_i^* \phi_k$ approaches 0 if and only if $\phi_k + \ker A_i^*$ approaches $\ker A_i^*$.

PROOF. Trivial modifications of a computation found in Green and Krall [4] show that if $y \in D$ and $z \in D^+$, then

$$(Ly, z) - (y, L^+z) = \int_0^1 [z^*(Ly) - (L^+z)^*y]\, ds$$
$$= -\phi^*(z)\left[\sum_{i=0}^{\infty} A_i y(t_i) + \int_0^1 Ky\, ds\right] = 0.$$

Thus $L \subset (L^+)^*$.

To show the reverse inclusion, let $z \in D_0^+$. Then, since $D_0^+ \subset D^+$ and $\phi(z)=0$, we find $L_0^+ \subset L^+$. This implies $(L^+)^* \subset (L_0^+)^* = L'$. Hence the domain of $(L^+)^*$ is contained in D.

For arbitrary $z \in D^+$, the calculation of the first part of the proof shows

$$\phi^*(z)\left[\sum_{i=0}^{\infty} A_i y(t_i) + \int_0^1 Ky\, ds\right] = 0.$$

Since the range of ϕ is R^m, it is the term in brackets which vanishes. Thus the domain of $(L^+)^*$ satisfies the boundary condition and is in D. Therefore $(L^+)^* \subset L$, and the two are equal.

THEOREM 3. $L^* = L^+$.

PROOF. Since L^+ is closed by Theorem 1, $(L^+)^{**} = L^+$. But since $(L^+)^* = L$ by Theorem 2, we find $L^* = (L^+)^{**} = L^+$.

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