ADJOINTS OF MULTIPOINT-INTEGRAL BOUNDARY VALUE PROBLEMS

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Abstract. The dual system to \( L y = y' + P y, \)
\[
\sum_{i=0}^{\infty} A_i y(t_i) + \int_0^1 K(t) y(t) \, dt = 0
\]
is found when the setting is \( L^p_{\infty}(0, 1), 1 < p < \infty. \)

Introduction. Let \( X \) denote the Banach space \( L^p_{\infty}(0, 1) \), consisting of \( n \)-dimensional vectors under the norm
\[
\| x \| = \left( \int_0^1 \left\{ \sum_{i=1}^{n} |x_i(t)|^2 \right\}^{p/2} \, dt \right)^{1/p},
\]
\( 1 < p < \infty. \) In \( X \) and \( X^* (= L^p_{\infty}(0, 1), 1/p + 1/q = 1) \) let us define the following subspaces:

(a) Let \( D' \) denote those vectors in \( X \) which are absolutely continuous.
(b) Let \( D^+_q \) denote those vectors in \( X^* \) which are absolutely continuous and vanish at \( t=0, \ t=1. \)
(c) Let \( A_i, i=0, \ldots, \) be \( m \times n \) constant matrices satisfying \( \sum_{i=0}^{\infty} \| A_i \| < \infty \) (\( \| \cdot \| \) here denotes a convenient norm) and \( \bigcap_{i=0}^{\infty} \ker A^*_i = 0. \) Let \( \{ t_i \}_{i=0}^{\infty} \) be a collection of points in \( [0, 1] \) \( (t_0=0, \ t_1=1) \), and let \( K(t) \) be an \( m \times n \) matrix valued function whose rows are in \( X^*. \) For all \( y \) in \( D' \) we define the (discontinuous) boundary functional \( V \) by
\[
Vy = \sum_{i=0}^{\infty} A_i y(t_i) + \int_0^1 K(t) y(t) \, dt,
\]
and denote by \( D \) the kernel of this functional. \( Vy=0 \) for all \( y \in D. \)
(d) Let \( D^+_z \) denote those vectors \( z \) in \( X^* \) for which there exists an \( m \times 1 \) matrix valued functional \( \phi(z) \) such that
(1) \( z(t) + \sum_{i=0}^{\infty} A_i^* \phi(z) \lambda(t_i, 1) \) is absolutely continuous on \([0, 1]. \)
(\( \lambda(t_i, 1) \) is the characteristic function of \( (t_i, 1). \))

Presented to the Society, January 18, 1972; received by the editors November 1, 1971 and, in revised form, March 20, 1972.

AMS (MOS) subject classifications (1970). Primary 34B05, 34B10, 34B25; Secondary 44A60.

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(2) If \( P(t) \) is a continuous \( n \times n \) matrix valued function, then the expression

\[
I^+z = -z' + P^*z + K^*\phi(z)
\]

exists a.e. and is in \( X^* \).

We note that \( \phi \) is uniquely defined, for if two such \( \phi, \phi_1 \) and \( \phi_2 \), exist, then by subtraction \( \sum_{t_0}^{t_1} A^*_t(\phi_1 - \phi_2)\lambda(t, 1) \) would be absolutely continuous. This is impossible. Further note that the range of \( \phi \) is all of \( R^n \), that \( D^+_0 \subset D^+ \), and that \( \phi(z) = 0 \) if and only if \( z \in D^+_0 \). Hence \( D^+ \) is non-empty.

Finally, letting \( ly = y + Py \), \( l^+z \) as given above, we define the following operators:

(a) \( L' \) is given by \( L'y = ly \) for all \( y \) in \( D' \).
(b) \( L^+_0 \) is given by \( L^+_0z = l^+z \) for all \( z \) in \( D^+_0 \).
(c) \( L \) is given by \( Ly = ly \) for all \( y \) in \( D \).
(d) \( L^+ \) is given by \( L^+z = l^+z \) a.e. for all \( z \) in \( D^+ \).

Our principal result is that \( L \) and \( L^+ \) are dual operators. Brown [1] has previously shown that \( D \) is dense in \( X \), so \( L^* \) exists. Further Bryan [2] used \( L^+ \) as the basis of a definition of \( L^* \). The result is well known under the classical endpoint conditions. However, because of the possible density of the boundary points \( \{t_i\} \) in the interval \( [0, 1] \), the techniques now used are different from those used previously.

The result also generalizes similar results recently obtained by Green and Krall [4] and Krall [5].

The closure of \( L^+ \).

**Theorem 1.** The operator \( L^+ \), defined by \( L^+z = -z' + P^*z + K^*\phi(z) \), for all \( z \in D^+ \), is closed.

**Proof.** We recall that the operator \( L^+ \) is closed if when \( \lim_{k \to \infty} z_k = z \) (\( z_k \in D^+ \)), and \( \lim_{k \to \infty} L^+z_k = y \), then \( z \in D^+ \), and \( L^+z = y \).

Let \( \varepsilon(s) = \sum_{t_0}^{t_1} A^*_t\lambda(t, 1)(s) \). Then

\[
zk(s) - z_l(s) = \int_0^s (z'_k - z'_l) \, d\xi + \varepsilon(s)\phi(z_l - z_k)
\]

\[
= \int_0^s L^+(z_l - z_k) \, d\xi + \int_0^s P^*(z_k - z_l) \, d\xi
\]

\[
+ \left[ \int_0^s K^* \, d\xi - \varepsilon(s) \right] \phi(z_k - z_l).
\]

Since

\[
\left\| \int_0^s L^+(z_l - z_k) \, d\xi \right\|_{X^*} \leq \left\| L^+(z_l - z_k) \right\|_{X^*}
\]

and

\[
\left\| \int_0^s P^*(z_k - z_l) \, d\xi \right\|_{X^*} \leq \left\| P^* \right\| \left\| z_k - z_l \right\|_{X^*}
\]
by Hölder's inequality, we have
\[ \left\| \int_0^s K^* d\xi - \epsilon(s) \right\|_{\mathcal{X}^*} \leq \|z_k - z_l\|_{\mathcal{X}^*} + \|P^*\| \|z_l - z_k\|_{\mathcal{X}^*} + \|L^*(z_l - z_k)\|_{\mathcal{X}^*}. \]

By assumption, each of the terms on the right approaches 0 as \( k, l \) approach \( \infty \). Therefore the functions
\[ F_k(s) = \left( \int_0^s K^* d\xi - \epsilon(s) \right) \phi(z_k) \]
converge in measure. Hence there is a subsequence \( F_{k_l}(s) \) which converges almost everywhere. Letting \( s \) approach \( t \) from above and below implies \( F_{k_i}(t_i) - F_{k_j}(t_j) = A^*_i \phi_{k_i} \) converges. Therefore \( \phi_{k_i} - \phi_{k_j} \), approaches the kernel of \( A^*_i \) as \( k_i, k_j \) approach \( \infty \). Since this holds for all \( i \), and \( \bigcap_{i=0}^\infty \text{ker } A^*_i = 0 \), we conclude that \( \phi_{k_i} - \phi_{k_j} \) approaches 0. In other words \( \phi_{k_i} \) converges.

Since
\[ z_k(s) = -\int_0^s L^+ z_{k_i} d\xi + \int_0^s P^* z_{k_i} d\xi + \left[ \int_0^s K^* d\xi - \epsilon(s) \right] \phi(z_k) \]
and \( \phi(z_k) \) converges, we may take limits to find
\[ z(s) = -\int_0^s y d\xi + \int_0^s P^* z d\xi + \left[ \int_0^s K^* d\xi - \epsilon(s) \right] \phi, \]
or
\[ z(s) + \epsilon(s) \phi = -\int_0^s y d\xi + \int_0^s P^* z d\xi + \int_0^s K^* d\xi \phi. \]

Since the right is absolutely continuous, \( z \) satisfies the first requirement of elements in \( D^+ \). Differentiating, we find \( z' = -y + P^* z + K^* \phi \), or \( y = l^+ z \) a.e., the second condition.

The duals of \( L \) and \( L^+ \). Let us denote the dual of an operator \( A \) by \( A^* \).

**Lemma.** \((L^+_0)^* = L'\).

**Proof.** This is well known. See Goldberg [3, p. 51].

**Theorem 2.** \((L^+)^* = L\).

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1 The matrix \( A^*_i \) defines an isomorphism \( f_i \) from the factor space \( R^m/\ker A^*_i \) onto the range of \( A^*_i \). Thus \( A^*_i \phi_k \) approaches 0 if and only if \( \phi_k + \ker A^*_i \) approaches \( \ker A^*_i \).
PROOF. Trivial modifications of a computation found in Green and Krall [4] show that if \( y \in D \) and \( z \in D^+ \), then

\[
(Ly, z) - (y, L^+z) = \int_0^1 [z^*(Ly) - (L^+z)^*y] \, ds
\]

\[
= -\phi^*(z) \left[ \sum_{i=0}^{\infty} A_i y(t_i) + \int_0^1 Ky \, ds \right] = 0.
\]

Thus \( L \subseteq (L^+)^* \).

To show the reverse inclusion, let \( z \in D_0^+ \). Then, since \( D_0^+ \subseteq D^+ \) and \( \phi(z) = 0 \), we find \( L_0^+ \subseteq L^+ \). This implies \( (L^+)^* \subseteq (L_0^+)^* = L' \). Hence the domain of \( (L^+)^* \) is contained in \( D \).

For arbitrary \( z \in D^+ \), the calculation of the first part of the proof shows

\[
\phi^*(z) \left[ \sum_{i=0}^{\infty} A_i y(t_i) + \int_0^1 Ky \, ds \right] = 0.
\]

Since the range of \( \phi \) is \( \mathbb{R}^m \), it is the term in brackets which vanishes. Thus the domain of \( (L^+)^* \) satisfies the boundary condition and is in \( D \). Therefore \( (L^+)^* \subseteq L \), and the two are equal.

**Theorem 3.** \( L^* = L^+ \).

**Proof.** Since \( L^+ \) is closed by Theorem 1, \( (L^+)^** = L^+ \). But since \( (L^+)^* = L \) by Theorem 2, we find \( L^* = (L^+)^** = L^+ \).

**References**


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