A MODEL FOR CLOSED ORIENTABLE 3-MANIFOLDS OF GENUS 1

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Abstract. In this paper we obtain a rectangular model for each closed orientable 3-manifold of genus 1 by identifications of the unit cube via unimodular plane involutions. The desirability of the model is indicated with regard to the fibering of such manifolds by circles, together with the development of a corresponding model for the connected sum of two such manifolds.

1. Introduction. In [1], a “rectangular” model \( \mathcal{X} \) for \( S^3 \) is obtained from the unit cube via certain boundary identifications and is used to show how the \( S^1 \)-fibrings (Seifert fibrings) of \( S^3 \) arise in a natural way from the Hilbert modular functions. Here the main emphasis is on the fibrings, and the author quite naturally chooses to show that the closed (compact, without boundary) orientable 3-manifold \( \mathcal{X} \) is in fact \( S^3 \) by observing that \( \mathcal{X} \) has an \( S^1 \)-fibring with base space \( S^2 \). The need to generalize \( \mathcal{X} \) to include the lens spaces \( L(p, q) \) is suggested in [1] and is therefore the motivation for this paper.

It is well known that each closed orientable 3-manifold \( M \) of genus 1 is either a lens space or the duplication \( S^2 \times S^1 \). The purpose of this paper is to obtain the corresponding model \( \mathcal{X} \) for each such \( M \) from the unit cube by identifying coordinates \( x, y \) each modulo 1 and the remaining faces each via a unimodular plane involution. An indication as to the desirability of the model for visualizing the \( S^1 \)-fibrings of \( M \) is included in the final section.

2. Remarks on unimodular plane involutions. By unimodular plane involution is meant a linear operator \( T_0 \) of the plane with integral coefficients which is involutory, and whose coefficients have determinant \(-1\). In general, \( T_0 \) is of the form

\[
T_0(x, y) = (a_0x + b_0y, c_0x - a_0y).
\]

A linear form \( s_0 \) which is invariant under \( T_0 \) is given by \( s_0 = (a_0 + 1)x + b_0y \) (or \( s_0 = c_0x + 2y \) in case \( a_0 + 1 = b_0 = 0 \)), and if \( d_0 \) is the gcd of the coefficients

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of \( s_0 \), then \( S_0 = s_0/d_0 \) is a generator of the submodule of \( T_0 \)-invariant linear forms with integral coefficients. Similarly, a linear form \( s'_0 \) which is anti-invariant under \( T_0 \) is given by \( s'_0 = (a_0 - 1)x + b_0y \) (or \( s'_0 = c_0x - 2y \) in case \( a_0 - 1 = b_0 = 0 \)), and if \( d'_0 \) is the gcd of its coefficients, then \( S'_0 = s'_0/d'_0 \) is a generator of the submodule of \( T_0 \)-anti-invariant linear forms.

Let \( \delta_0 = \det(S_0, S'_0) \) denote the determinant of the coefficients of \( S_0 \) and \( S'_0 \). By choosing the sign of \( S_0 \) accordingly, \( \delta_0 \) can be taken to be positive, and by direct calculation, \( \delta_0 = 2 \) or \( 1 \), depending on whether \( d'_0 \) divides or does not divide \( b_0/d_0 \).

Let \( S_0 \) denote \( S'_0 \) whenever \( \delta_0 = 1 \) and \( (S_0 + S'_0)/2 \) otherwise. In either case, a change of coordinates to the new coordinates \( S_0, S'_0 \) is unimodular, and (replacing \( S_0, S'_0 \) by \( x, y \)) we have

**Lemma 1.** By a suitable change of coordinates, each unimodular plane involution can be written in the form

\[
T_0(x, y) = (x, (\delta_0 - 1)x - y), \quad \text{for } \delta_0 = 1 \text{ or } 2.
\]

Since the involutory property of \( T_0 \) is invariant under the identification of the plane which identifies coordinates \( x, y \) each modulo 1, then \( T_0 \) induces a “folding” of the unit square which identifies each point of the square with its image under \( T_0 \), modulo the square. By a change of coordinates (with new coordinates \( S_0, S'_0 \)), write \( T_0 \) in the form of Lemma 1. Then for \( \delta_0 = 1 \), \( T_0(x, y) = (x, -y) \) with invariant and anti-invariant linear forms \( S_0 = x \) and \( S'_0 = y \) respectively, so that \( T_0 \) has invariant lines \( x = \text{const.} \) and anti-invariant lines \( y = \text{const.} \). \( (T_0 \) maps the line \( x = c \) onto itself and the line \( y = c \) onto \( y = -c \).) For \( \delta_0 = 2 \), \( T_0(x, y) = (x, x - y) \) with \( S_0 = x \) and \( S'_0 = -x + 2y \), so that \( T_0 \) has invariant lines \( x = \text{const.} \) and anti-invariant lines \( -x + 2y = \text{const.} \). In either case, reinserting \( S_0, S'_0 \) for \( x, y \) we have

**Lemma 2.** \( T_0 \) induces a folding of the unit square which identifies points along the lines \( S_0 = \text{const.} \) and symmetrically through the ramification line \( S_0 = 0 \), with coordinates \( x, y \) each taken modulo 1.

Let \( V \) be the half-cube \( \{0 \leq x, y \leq 1, 0 \leq z \leq 1/2 \} \), and \( V_0 \) the compact orientable 3-manifold obtained from \( V \) by identifying coordinates \( x, y \) each modulo 1, and the face \( z = 1/2 \) via \( T_0 \) as in Lemma 2. That \( V_0 \) is a toroid (solid torus of genus 1) with boundary \( \pi_0 \) (the \( z = 0 \) face of \( V \)) is apparent by changing coordinates to \( S_0, S'_0 \). Then \( T_0 \) takes the form of Lemma 1, and in either case \( (\delta_0 = 1 \text{ or } 2) \) the simple closed curve \( x = 0 \) on \( \pi_0 \) is nullhomotopic within \( V_0 \). Moreover, the required homotopy, given by \( H_\tau(0, y, z) = (0, y, z - \tau(z - 1/2)) \), deforms the \( x = 0 \) face of \( V_0 \) to a point, since \( H_1(0, y, z) = (0, y, 1/2) \) is self-cancelling under \( T_0 \). Hence the \( x = 0 \)
face of $V_0$ is topologically a 2-disc, and since $V_0$ is the product of its $x=0$ face and the simple closed curve $y=0$ on $\pi_0$, $V_0$ is a toroid. So, in terms of $S_0$, $S_0$ we have

**Lemma 3.** $V_0$ is a toroid with meridian $S_0=0$ and longitude $S_0=0$.

3. **Genus 1 model.** If $M$ is the closed orientable 3-manifold of genus 1 obtained from toroids $V_0$ and $V_1$ by identifying their boundaries according to an orientation reversing homeomorphism, and if $V_i$ has meridian $M_i$ and longitude $B_i$, $i=0, 1$, then if $M_i \sim \pm M_0$ ($\sim$ means homotopic) on the common boundary $\pi$, $M$ is clearly the duplication $S^2 \times S^1$. If $M_i \sim pB_0 + qM_0$ on $\pi$, with $p>0$ and $q$ relatively prime integers, $M$ is the lens space $L(p, q)$ [3, p. 554]. Let $T_0$ and $T_1$ be unimodular plane involutions with $T_0$ in the form of Lemma 1. Denote by $\mathcal{H}_1$ the closed orientable 3-manifold obtained from the unit cube by identifying coordinates $x, y$ each modulo 1 and the faces $z=0$ and $z=1$. The plane $z=1/2$ decomposes $\mathcal{H}_1$ into toroids $V_0$ and $V_1$ with common boundary $\pi$ (the $z=1/2$ cross-section), so that $\mathcal{H}_1$ is a genus 1 manifold.

By Lemma 3, $V_0$ has meridian $x=0$ and longitude $y=0$ on $\pi$. If $b_i \neq 0$, the invariant linear form for $T_1$ is given by $S_1 = [(a_1 + 1)x + b_1 y]/d_1$, so that $V_1$ has meridian $(a_1 + 1)/d_1 x + b_1 y = 0$ on $\pi$. If $b_1 = 0$, then $a_1 = \pm 1$. For $a_1 = -1$, $S_1 = (c_1 x + 2y)/d_1$, so that $V_1$ has meridian $c_1 x + 2 y = 0$ on $\pi$, and for $a_1 = +1$, $T_1(x, y) = (x, c_1 x - y)$ with $S_1 = x$, so that $V_1$ has meridian $x=0$ on $\pi$. Combining the above remarks we have

**Theorem.** If $p \geq 0$ and $q$ are relatively prime integers and $T_1$ a unimodular plane involution with invariant linear form $S_1 = qx + py$, then $\mathcal{H}_1$ is the duplication $S^2 \times S^1$ whenever $p=0$ and the lens space $L(p, q)$ otherwise.

A partial converse can be given by

**Corollary 1.** Each lens space $L(p, q)$ has a corresponding model $\mathcal{H}_1$, where $T_0$ is given by $T_0(x, y) = (x, -y)$ and $T_1$ by $T_1(x, y) = ((dq - 1)x + dp, cqx - (dq - 1)y)$, with $c, d$ integral solutions to $cp + dq = 2$.

Let $T_0$ and $T_1$ be arbitrary unimodular plane involutions, and $\mathcal{H}$ be obtained from the unit cube via $T_0$ and $T_1$ as before. We say that $T_0$ and $T_1$ are $p$-compatible if $\det(S_0, S_1) = p \geq 0$. ($p \geq 0$ is guaranteed by choosing the sign of $S_1$ appropriately.) Since $p$-compatibility is invariant under unimodular change of coordinates, we have

**Corollary 2.** If $T_0$ and $T_1$ are 0-compatible, $\mathcal{H} = S^2 \times S^1$, and if $p$-compatible ($p>0$), $\mathcal{H} = L(p, q)$ for some $q$.

By direct calculation we have

**Corollary 3.** If $\mathcal{H} = L(p, q)$, then $q$ can be chosen to be $q = \det(S_1, S_0)$. 

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4. Remarks. Each closed orientable 3-manifold of genus 1 is a Seifert fibre space with base space $S^2$ and at most 2 singular fibres. In the model $\mathcal{N}$, the singular fibres can be taken to be $S^0_0=0$ and $S^1_1=0$ on the faces $z=0$ and $z=1$ respectively. Each fibering of $\mathcal{N}$ is then induced by a simple closed curve (fibre) $H=rx+sy$ which is nonhomotopic with $S_0$ and $S_1$ on the $z=1/2$ cross-section of $\mathcal{N}$, as these forms are self-cancelling under $T_0$ and $T_1$ respectively. Viewing $L(p,q)$ in the simplified form of Corollary 1, the singular fibre invariants of Seifert are readily obtained [2, p. 181].

A second perhaps pleasing visualization arises when considering the connected sum $M \# M'$ of two closed orientable 3-manifolds of genus 1, obtained by removing the interior of a 3-cell from each and then identifying the resulting boundaries via an orientation reversing homeomorphism. If $\mathcal{N}$ and $\mathcal{N}'$ are the models arising from $T_0$, $T_1$ and $T'_0$, $T'_1$ respectively, then $\mathcal{N} \# \mathcal{N}'$ can be obtained from $\mathcal{N}$ by removing the interior of a cube from its interior and then identifying the side faces of the resulting boundary straight across, the upper face via $T'_0$ and the lower face via $T'_1$. (Reversing the orientation of $\mathcal{N}$ is equivalent to interchanging the roles of $T_0'$ and $T_1'$.) If $\mathcal{N}$ and $\mathcal{N}'$ are written in the simplified form of Corollary 1, we observe by performing the appropriate identifications on $\mathcal{N}'$, that when $\mathcal{N}'=S^2 \times S^1$, then summing $\mathcal{N}'$ to $\mathcal{N}$ does correspond to the usual operation of adding a handle (removing the interiors of two disjoint 3-cells from $\mathcal{N}$ and then identifying the resulting boundaries by an orientation reversing homeomorphism). On the other hand, if $\mathcal{N}'=S^3$, then the “inner” boundary identifies to a point and $\mathcal{N} \# \mathcal{N}' = \mathcal{N}$, as is expected. Furthermore, the toroid decomposition of $M \# M'$ into two solid toroids of genus 2 is apparent by cutting $\mathcal{N}' \# \mathcal{N}$ with the plane $z=1/2$.

References