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ON A QUESTION OF M. HOCHSTER

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ABSTRACT. It is shown that not all finite integral extensions of a Noetherian integrally closed domain A can be obtained by a process of adjoining integral elements whose monic irreducible equations over the quotient field have all the coefficients in the base ring and passing to A-subalgebras of the algebras so obtained.

Let A be a domain and A^* its quotient field. Let K be an algebraic closure of A^*. Let $\mathscr{C}(A)$ be the least class of extensions of A in K, closed under the two operations:

(a) passing to an A-subalgebra,

(b) adjoining an element whose monic irreducible equation over the quotient field of the domain has all the coefficients in the domain.

In [**H**], M. Hochster asks whether every finitely generated extension C of A in K, such that C is a finitely generated A-module and $C \to A^* = A$, must be in $\mathscr{C}(A)$, or even whether, if A is normal, every finitely generated integral extension of A in K must be in $\mathscr{C}(A)$.

We give here an example of a Noetherian unique factorization domain A and a finitely generated integral extension which does not belong to $\mathscr{C}(A)$.

For this, we first prove the following obvious

LEMMA. *If $C \in \mathscr{C}(A)$, then the A-module inclusion $A \hookrightarrow C$ is A-split.*

PROOF. If C is obtained from an extension B of A such that $A \hookrightarrow B$ is A-split, then, by applying either (a) or (b), $A \hookrightarrow C$ is split. For, if C were obtained from B by (a), the restriction of the splitting $B \to A$ to C gives the splitting $C \to A$. On the other hand, if C is obtained from B by (b), C is a free B algebra and the inclusion $B \to C$ is B-split. By composing we get a splitting $C \to A$. Now, by induction on the number of times the operations (a) and (b) are used, the lemma follows.

Let k be a field of characteristic 2. Let $R = k[X_1 \cdots X_4]$ be the polynomial ring in 4 variables over k. Then $SL(4, k)$ acts on R naturally. Let G

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be the subgroup of $SL(n, k)$ generated by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then G is a cyclic group of 4 elements.

Let A be the invariants in R under G. By the proposition in [**B**, p. 655], A is a unique factorization domain which is not Cohen-Macaulay.

PROPOSITION. *With the notation as above, the domain R is a finite integral extension of A and R is not in $\mathscr{C}(A)$.*

PROOF. That R is a finite integral extension of A is clear. We prove the inclusion $A \hookrightarrow R$ is not A-split. Since R is finite integral over A, $\operatorname{depth}_R(R) = \operatorname{depth}_A(R)$. But if A were an A-direct summand of R, then $\operatorname{depth}_A R = \operatorname{depth}_A A$. But, since A is not Cohen-Macaulay, $\operatorname{depth}_A A \lneqq \dim A = 4$. Also $\operatorname{depth}_R R = 4$, a contradiction to the assumption $A \hookrightarrow R$ is A-split.

REFERENCES

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