ABSOLUTELY STRUCTURALLY STABLE DIFFEOMORPHISMS

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Abstract. This paper gives a proof that if a diffeomorphism is structurally stable in a strong sense then it satisfies Axiom A of S. Smale. This provides a weakened converse of a theorem of J. Robbin on structural stability.

In this note we prove some results on structurally stable diffeomorphisms analogous to recent results on $\Omega$-stability in [2] and [4]. Recall that a diffeomorphism $f: M \to M$ of a compact manifold is said to be structurally stable if there is a neighborhood $N$ of $f$ in $\text{Diff}^1(M)$ with the property that to each $g \in N$ there corresponds a homeomorphism $h$ of $M$ such that $g \circ h = h \circ f$. In [5], J. Robbin proves a conjecture of Smale which provides sufficient conditions for $f$ to be structurally stable when $f$ is $C^2$. In this paper we show that if the definition of structurally stable is strengthened these conditions are necessary as well as sufficient.

Definition. A diffeomorphism $f: M \to M$ is absolutely structurally stable if there is a neighborhood $N$ of $f$ in $\text{Diff}^1(M)$ and a function $\phi: N \to C^0(M,M)$ such that:

1. $\phi(g)$ is a homeomorphism for each $g \in N$, and $\phi(f) = \text{id}: M \to M$.
2. $g \circ \phi(g) = \phi(g) \circ f$.
3. There is a constant $K > 0$ such that

$$\sup_{x \in M} d(\phi(g)(x), x) \leq K \sup_{x \in M} d(f(x), g(x)),$$

where $d$ is a metric on $M$.

Theorem 1. If $f: M \to M$ is $C^2$ and $M$ is compact then $f$ is absolutely structurally stable if and only if $f$ satisfies Axiom A and the strong transversality property.

We remark that the requirement that $f$ be $C^2$ is necessary for only one direction of the implication. Even if $f$ is only $C^1$ it is still true that absolute structural stability implies Axiom A and the strong transversality property.
In [5], Robbin proves that if \( f \) is \( C^2 \) and satisfies Axiom A and the strong transversality property then a function \( \phi \) satisfying (1) and (2) of the definition above exists and is continuous. However, a stronger statement holds.

**Theorem 2.** If \( f \) is \( C^2 \) and is absolutely structurally stable then \( \phi \) and \( N \) can be chosen so that \( \phi : N \to C^0(M, M) \) is a \( C^1 \) map.

**Proof of Theorem 1.** We first show that Axiom A and strong transversality imply absolute structural stability. The continuous maps from \( M \) to itself, \( C^0(M, M) \), have the structure of a \( C^\infty \) Banach manifold. We will identify the tangent space to \( C^0(M, M) \) at \( \text{id} \) with \( \Gamma^0 \), the space of continuous sections of the tangent bundle \( TM \) (see [1] or [3] for this). \( \Gamma^0 \) is a Banach space with the sup norm.

Let \( \exp : TM \to M \) be the exponential map arising from a \( C^\infty \) Riemannian metric on \( M \); then there is an open neighborhood \( U \) of \( \text{id} \) in \( \Gamma^0 \) such that the map \( \Psi : U \to C^0(M, M) \) given by \( \Psi(\gamma) = \exp \circ \gamma \) defines a \( C^\infty \) chart on \( C^0(M, M) \). Note that if \( h = \Psi(\gamma) \) then \( \sup_{x \in M} d(h(x), x) = \| \gamma \| \), where \( \| \cdot \| \) is the sup norm on \( \Gamma^0 \) and also \( D\Psi(0) : \Gamma^0 \to \Gamma^0 \) is the identity.

We define the adjoint map \( f^# : \Gamma^0 \to \Gamma^0 \) by \( f^#(\gamma) = df^{-1} \circ \gamma \circ f \) (this coincides with the definition of \( f^# \) in [5] but is the inverse of the \( f^# \) defined in [2]).

In [5], Robbin shows that if \( f \) is \( C^2 \) and satisfies Axiom A and the strong transversality condition then there is a neighborhood \( N \) of \( f \) in \( \text{Diff}^3(M) \), and a linear map \( J : \Gamma^0 \to \Gamma^0 \) satisfying

1. \( (I - f^#) \circ J = I \), the identity on \( \Gamma^0 \), and
2. for each \( g \in N \) there is a unique point, \( \hat{\phi}(g) \), in \( U \cap \text{image } J \) which has the property that if \( h = \Psi(\hat{\phi}(g)) = \exp \circ \hat{\phi}(g) \) then \( h \) is a homeomorphism and \( g^{-1} \circ h \circ f = h \). Also \( \hat{\phi} : N \to \Gamma^0 \) is continuous.

If we define \( \phi(g) \) to be \( \Psi(\hat{\phi}(g)) = \exp \circ \hat{\phi}(g) \) then \( \phi \) satisfies (1) and (2) of the definition of absolute structural stability so we need only show that (3) holds.

Since \( (I - f^#) \circ J = I \), if \( R \) is the image of \( J \), \( R \) is a closed subspace of \( \Gamma^0 \) and is a complement to the kernel of \( (I - f^#) \). Also \( (I - f^#) : R \to \Gamma^0 \) is an isomorphism. Define \( F : C^0(M, M) \to C^0(M, M) \) by \( F(h) = f^{-1} \circ h \circ f \) and define \( \tilde{F} : U' \to \Gamma^0 \) by \( \tilde{F}(\gamma) = \Psi^{-1} \circ F \circ \Psi \) where \( U' \) is a suitably chosen neighborhood of \( 0 \) in \( U \). Let \( W = R \cap U \), then \( (I - \tilde{F}) : W \to \Gamma^0 \) is a \( C^1 \) map and its derivative at \( 0 \) is \( (I - f^#) : R \to \Gamma^0 \) (for this see [3] or [1, p. 780]). Since \( (I - f^#) : R \to \Gamma^0 \) is an isomorphism by the inverse function theorem there is a neighborhood \( W' \) of \( 0 \) in \( W \) on which \( I - \tilde{F} \) is a diffeomorphism. Hence there is a constant \( q > 0 \) such that \( \| (I - \tilde{F})(\gamma) \| \geq q \| \gamma \| \) for all \( \gamma \in W' \), so \( \| \gamma \| \leq q^{-1} \| \gamma - \tilde{F}(\gamma) \| \). Since exp is \( C^2 \) there is a constant \( K_1 > 0 \) such that \( \| \gamma - \tilde{F}(\gamma) \| \leq K_1 \sup_{x \in M} d(\exp(\gamma(x)), \exp(\tilde{F}(\gamma)(x))) \) if \( \| \gamma \| \) is sufficiently
small. Hence if $N$ is a sufficiently small neighborhood of $f$ in Diff$(M)$, $g \in N$, and we let $h = \phi(g)$ and $\gamma = \Psi^{-1}(\phi(g))$ we have

$$
\sup_{x \in M} d(h(x), x) = \|\gamma\| \leq q^{-1} \|\gamma - \hat{F}(\gamma)\|
$$

$$
= K_1 q^{-1} \sup_{x \in M} d(F(h)(x), h(x))
$$

$$
= K_1 q^{-1} \sup_{x \in M} d(f^{-1} \circ h \circ f(x), h(x))
$$

$$
= K_1 q^{-1} \sup_{x \in M} d(f^{-1} \circ h \circ f(x), g^{-1} \circ h \circ f(x))
$$

$$
= K_1 q^{-1} \sup_{x \in M} d(f^{-1}(x), g^{-1}(x)).
$$

But $\sup_{x \in M} d(f^{-1}(x), g^{-1}(x)) = \sup_{x \in M} d(f^{-1} \circ g(x), x)$ and since $f^{-1}$ is $C^1$ there is a constant $K_2$ such that $d(x, y) \leq K_2 d(f(x), f(y))$ for all $x, y \in M$. Thus if $K = K_1 K_2 q^{-1}$,

$$
\sup_{x \in M} d(h(x), x) \leq K \sup_{x \in M} d(g(x), f(x))
$$

as was to be shown.

To complete the proof of Theorem 1 we note that if $f$ is absolutely structurally stable it is a fortiori absolutely $\Omega$-stable so by the results of [2] and [4] it satisfies Axiom A and by a theorem asserted in [6] which is not difficult to prove, a structurally stable diffeomorphism which satisfies Axiom A also satisfies the strong transversality condition. Q.E.D.

**Proof of Theorem 2.** We use the same terminology as in the proof of Theorem 1. By Theorem 1, $f$ and hence also $f^{-1}$ will satisfy Axiom A and the strong transversality property so we can again cite the results of Robbin [5], this time applied to $f^{-1}$. Namely, there is a neighborhood $N'$ of $f^{-1}$ in Diff$(M)$ and a linear map $J: \Gamma^0 \to \Gamma^0$ such that $(I - f^{-1} \circ f)(x) = J$ and if $g \in N'$ there is a unique $\gamma \in R$, the image of $J$, such that $g \circ h \circ f^{-1} = h$ if $h = \exp \circ \gamma$.

Now let $K$ be the kernel of $I - f^{-1}$ and recall that $\Gamma^0 = K \oplus R$. If $N_0$ is a sufficiently small neighborhood of $f$ in Diff$(M)$ then the map $H: N_0 \times K \times R \to \Gamma^0$, given by $H(g, \gamma_1, \gamma_2) = \gamma_2 - \Psi^{-1}(g \circ \Psi(\gamma_1, \gamma_2) \circ f^{-1})$ is well defined when $\|\gamma_1\|$ and $\|\gamma_2\|$ are small enough. $\Psi$ is a $C^\infty$ chart and the map $C^0(M, M) \to C^0(M, M)$ which sends $(g, h)$ to $g \circ h \circ f^{-1}$ is $C^1$ (see [1]), so $H$ is a $C^1$ map. It also follows from the results of [1] that the partial $D_h H(f, 0, 0): R \to \Gamma^0$ is just $I - f^{-1}$ which is an isomorphism since $J$ is its inverse. Hence by the implicit function theorem there is a neighborhood $V$ of $(f, 0)$ in $N_0 \times K$ and a unique $C^1$ map $\theta: V \to R$ such that $H(g, \gamma_1, \theta(\gamma_1, \gamma_1)) = 0$ for all $(g, \gamma_1) \in N_0 \times K$. Let $N = \{g|(g, 0) \in V\}$ and let $\hat{\phi}: N \to R$ be given
by \( \hat{\phi}(g) = \theta(g, 0) \), then \( \hat{\phi} \) is \( C^1 \) and \( H(g, 0, \hat{\phi}(g)) = 0 \) so \( \hat{\phi}(g) = \Psi^{-1}(g \circ \Psi(\hat{\phi}(g)) \circ f^{-1}) \). But if \( \gamma \) is section corresponding to \( g^{-1} \) which is guaranteed by the result of Robbin cited above then \( \gamma \in R \) and \( \gamma = \Psi^{-1}(g \circ \Psi(\gamma) \circ f^{-1}) \) so by uniqueness of \( \hat{\phi}(g) \), \( \gamma = \hat{\phi}(g) \). Thus if we define \( \phi = \Psi \circ \hat{\phi} \), then \( \phi \) is a \( C^1 \) map but also if \( g \in N \), \( \phi(g) = \exp \circ \gamma = h \), a homeomorphism satisfying \( g \circ h \circ f^{-1} = h \). Q.E.D.

**BIBLIOGRAPHY**


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