ZARISKI’S THEOREM ON SEVERAL LINEAR SYSTEMS

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Abstract. We give a modern and fairly easy proof of (a slight improvement of) an important theorem of Zariski. The result gives conditions under which certain multigraded rings and modules associated with \( n \) linear systems are finitely generated, in a very strong sense.

Suppose \( L \) is a line bundle on a complete scheme \( X \) and \( R \) is a graded subring of \( \bigoplus_{n \geq 0} H^0(X, L^n) \) whose degree one part generates \( L \). Then \( \bigoplus_{n \geq 0} H^0(X, L^n) \) is a finitely generated \( R \) module. Zariski has given a very useful souped-up version of this fact, working with several line bundles simultaneously [1, 5.1]. Since his proof is difficult for newly educated geometers to follow, it seems worthwhile to give a modern proof. That is the only purpose of this paper.

Before we state our slight improvement of Zariski’s theorem, we must make some definitions. By an “\( m \)-fold graded ring,” we mean a ring \( G \) together with a direct sum decomposition \( G = \bigoplus \{ G_\alpha : \alpha \in \mathbb{Z}_m \} \) such that the multiplication map factors through maps \( G_\alpha \otimes G_\beta \to G_{\alpha + \beta} \). We let \( e_i \in \mathbb{Z}_m \) be the element with 1 in the \( i \)th place and zeroes elsewhere. Let \( G' \) be the sub-\( G_0 \) algebra of \( G \) generated by terms of total degree 1.

1. Definition. Let \( G \) be an \( m \)-fold graded ring, \( M \) a graded \( G \) module, and \( \alpha \) an integer between 1 and \( m \). Then \( M \) is “\( \alpha \)-finite” if for some integer \( n \), the maps \( G_\alpha \times M_\alpha \to M_{\alpha + e_i} \) are surjective whenever \( \alpha_i \geq n \). If \( M \) is \( \alpha \)-finite for all \( \alpha \), we say \( M \) is “polyfinite.”

2. Proposition. If \( M \) is finitely generated as a \( G' \) module, it is polyfinite. The converse holds if we assume that each \( M_\alpha \) is finitely generated as a \( G_0 \) module and that \( M_\alpha = 0 \) for any \( \alpha \ll 0 \).

Proof. First prove the following easy statements:

2.1 If \( M \) is \( i \)-finite, so is the shifted module \( M(\alpha) \) for any \( \alpha \in \mathbb{Z}_m \).

2.2 If \( M \) and \( N \) are \( i \)-finite, so is \( M \oplus N \).

2.3 A quotient of an \( i \)-finite module is \( i \)-finite.
Now it is clear that $G'$ is polyfinite as a module over itself, since all the maps $G'_{e_i} \times G'_{e_j} \to G'_{e_i+e_j}$ are surjective, if $\alpha_i \geq 0$. Moreover, any finitely generated graded $G'$ module $M$ is a quotient of a finite direct sum of modules $G'(\beta)$, one for each generator of degree $-\beta$. Thus, it is polyfinite as a $G'$ module and hence as a $G$ module.

To prove the converse, we see that if $M$ is polyfinite, there is a $\beta \in \mathbb{Z}^m$ such that the maps $G'_{e_i} \times M_{e_j} \to M_{e_i+e_j}$ are surjective if $\alpha_i \geq \beta_i$. Then we easily see that $\bigoplus_{e \leq \beta} M_e$ generates $M$ as a $G'$ module. Since $\bigoplus_{e \leq \beta} M_e$ is finitely generated as a $G_0$ module, it generates a finite $G'$ module, and the proof is complete. □

We can now state our version of Zariski’s theorem:

3. Theorem. Let $F$ be a coherent sheaf on a scheme $X$, proper over a field $k$, and let $L_1, \ldots, L_m$ be line bundles on $X$. Let $\Gamma$ be an $m$-fold graded subring of $\bigoplus \{ H^0(X, L_{1}^{\otimes e_1} \otimes \cdots \otimes L_{m}^{\otimes e_m}) : \alpha \in \mathbb{Z}^m \text{ and } \alpha \geq 0 \}$, and let $M$ be a graded $\Gamma$ submodule of $\bigoplus_{e \geq 0} H^n(X, F \otimes L_{1}^{\otimes e_1} \otimes \cdots \otimes L_{m}^{\otimes e_m})$. If the linear system $F(e_i)$ has no base points for each $i$, then $M$ is polyfinite.

Instead of proceeding directly with the proof of this theorem, we first consider what is essentially the universal case.

4. Proposition. Let $k$ be a field, $V_1, \ldots, V_m$ finite dimensional vector spaces over $k$, and $Z = \text{Spec } (V_1) \times \cdots \times (V_m)$. If $F$ is a coherent sheaf on $Z$ and if $\alpha \in \mathbb{Z}^m$, let $F(\alpha)$ be $F \otimes P_1^* (O_{\text{Spec } (V_1)}(\alpha_1)) \otimes \cdots \otimes P_m^* (O_{\text{Spec } (V_m)}(\alpha_m))$, where $p_i: Z \to \text{Spec } (V_i)$ is the projection. Then:

4.1 The natural map: $G = S'(V_1) \otimes \cdots \otimes S'(V_m) \to \bigoplus \alpha H^n(Z, O_Z(\alpha))$ is an isomorphism of $m$-fold graded rings.

4.2 $H^q(Z, O_Z(\alpha)) = 0$ if $q > 0$ and $\alpha \geq 0$.

4.3 $\bigoplus \{ H^q(Z, F(\alpha)) : \alpha \in \mathbb{Z}^m \text{ and } \alpha \geq 0 \}$ is a finitely generated $G$ module, for all $q$.

4.4 If $q > 0$, $H^q(Z, F(\alpha)) = 0$ for all $\alpha > 0$.

Proof. If $m=1$, this is Serre’s theorem [2, p. 47]. We shall prove 4.1, 4.2, and 4.3 by induction on $m$, where 4.3’ is the statement 4.3 for $F$ of the form $O_Z(\beta)$ for some $\beta \in \mathbb{Z}^m$. Assuming them proved for $m$ and for $Z$ with the same notation, we let $V$ be another vector space and prove them for $Z \times \text{Spec } (V)$. In the diagram, all the maps are the natural ones. If

$$
\begin{array}{ccc}
Z \times \text{Spec } (V) & \xrightarrow{g} & \text{Spec } (V) \\
\downarrow f & & \downarrow h \\
Z & \xrightarrow{p} & \text{Spec } k
\end{array}
$$

$\alpha \in \mathbb{Z}^m$ and $v \in Z$, then $O_{Z \times \text{Spec } (V)}(\alpha, v) = f_* O_Z(\alpha) \otimes g_* O_V(v)$. By the base change formula, the natural map: $O_Z(\alpha) \otimes R^q f_* g_* O_V(v) \to R^q f_* O_{Z \times \text{Spec } (V)}(\alpha, v)$
is an isomorphism. Since our diagram is Cartesian and $p$ is flat, the natural map: $p^*R^q\mathcal{O}_P(\nu)\to R^qf_*\mathcal{O}_P(\nu)$ is an isomorphism. Combining these with the base change formula for $p$, we get a natural isomorphism: $H^p(Z, O_Z(\alpha)) \otimes_k H^q(P, O_P(\nu)) \to H^p(Z, R^qf_*O_{Z\times P}(\alpha, \nu))$. By the induction hypothesis the map:

$$G_\mu \otimes S'(V) \to H^0(Z, O_Z(\alpha)) \otimes_k H^0(P, O_P(\nu)) \cong H^0(Z \times P, O_{Z\times P}(\alpha, \nu))$$

is an isomorphism, so 4.1 is proved. By induction, if $(\alpha, \nu) \geq 0$, we see that $H^p(Z, R^qf_*O_{Z\times P}(\alpha, \nu)) = 0$ if $p$ or $q > 0$, so by the Leray spectral sequence, $H^q(Z \times P, O_{Z\times P}(\alpha, \nu)) = 0$ if $i > 0$, and 4.2 is proved. Finally, for any $\beta$ and $\mu$, $\bigoplus_{(\alpha, \nu) \geq 0} H^p(\nu, O_{Z\times P}(\alpha, \nu))$ is finite as a $G$ module and

$$H^q(P, O_P(\mu + \nu))$$

is finite as an $S'(V)$ module, by the induction hypothesis; so their tensor product $\bigoplus_{(\alpha, \nu) \geq 0} H^p(Z, R^qf_*O_{Z\times P}(\beta + \alpha, \mu + \nu))$ is finite as a $G \otimes_k S'(V)$ module. Consequently the abutment $\bigoplus_{(\alpha, \nu) \geq 0} H^q(Z \times P, O_{Z\times P}(\beta + \alpha, \mu + \nu))$ is also finitely generated, so 4.3' is also proved.

To finish the proof, we recall that the Segre embedding [3, p. 93] shows that the sheaf $L = O_Z(1, \ldots, 1)$ is very ample on $Z$. Therefore any coherent $F$ on $Z$ is a quotient of a finite direct sum $E$ of copies of $L^i$, for some $v$. Moreover, 4.3 and 4.4 are proved for $E$, and also for all $q$ sufficiently large, since $H^q(Z, \nu. \nu, \ldots, \nu) = 0$ for $q > 0$. Now if $0 \to K \to E \to F \to 0$ is exact then we get exact sequences $H^q(Z, E(\alpha)) \to H^q(Z, F(\alpha)) \to H^{q+1}(Z, K(\alpha))$. Then the theorem for $E$ and a descending induction hypothesis on $E$ will imply our result for $F$. \(\Box\)

The proof of Theorem 3 is now quite easy. Let $V_i$ be the (finite dimensional) $k$ vector space $\Gamma_{e_i}$. Since $V_i$ has no basepoints, there is a map $f_i: X \to P(V_i)$ such that $f^*_i O_{P(V_i)}(1) = L_i$. Then if $f: X \to Z$ is the induced map, $f^*O_x(\alpha) = L^i_1 \otimes \cdots \otimes L^i_n = L^i$. Since $f$ is proper, the sheaves $R^qf_*\mathcal{F}$ are coherent on $Z$. Hence the $G$ module $\bigoplus_{(\alpha, \nu) \geq 0} H^q(Z, R^qf_*\mathcal{F}(\alpha))$ is finitely generated, so is the abutment $\bigoplus_{(\alpha, \nu) \geq 0} H^q(Z, F(\alpha))$. Since $G$ is noetherian, the $G$ submodule $M$ is also finitely generated. Finally, we note that $G_{e_i} = \Gamma_{e_i}$, so that by Proposition 2, $M$ is polyfinite as a $\Gamma$ module. This completes the proof. \(\Box\)

5. Corollary. Let $H$ be ample on a projective scheme $X$, let $L$ be a line bundle on $X$ generated by its global sections, and let $F$ be any coherent $O_X$ module. Then there exists an integer $J$ such that $H^q(X, F \otimes L^i \otimes H^n) = 0$ if $q > 0, i \geq 0$, and $j \geq J$.

Proof. Suppose $H^n$ is very ample, so that if $\Gamma = \bigoplus_{i,j} H^0(S, L^i \otimes H^n)$, $\Gamma$ satisfies the hypothesis of Theorem 3. We apply the theorem with
$F \otimes H^m$ in place of $F$, where $0 \leq m < n$, and conclude that each $\Gamma$ module
$\bigoplus_{i,j} H^q(X, F \otimes L^i \otimes H^{j+n+m})$ is 1-finite. Hence there exists an integer $I$, independent of $j$, such that the map:

$$H^0(X, L) \otimes H^q(X, F \otimes L^{i-1} \otimes H^{j+n+m}) \rightarrow H^q(X, F \otimes L^i \otimes H^{j+n+m})$$

is surjective if $i \geq I$, $j \geq 0$, and $0 \leq m < n$. Since $H^n$ is ample we can find $J$
such that $H^q(X, F \otimes L^i \otimes H^{j+n+m}) = 0$ if $q > 0$, $j \geq J$, $0 \leq m < n$, and $0 \leq i \leq I$, and it follows immediately by induction on $i$ that $H^q(X, F \otimes L^i \otimes H^j) = 0$
if $i \geq 0$ and $j \geq n(J+1)$.

**Remark.** Zariski has proved [1, 6.2] that if $H^0(X, L)$ has only finitely
many base points, then $H^0(X, L^i)$ has no base points for $i$ sufficiently
large, so we could weaken the hypothesis of the Corollary. His proof
makes essential use of Theorem 3, but since it is quite readable, I have
not included it here. I wish to thank the referee for filling a gap in my
proof of Corollary 5.

**References**

1. O. Zariski, *The theorem of Riemann-Roch for high multiples of an effective divisor
   MR 16, 953.

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