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PERTURBATIONS OF LINEAR m-ACCRETIVE OPERATORS

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ABSTRACT. A sufficient condition is given for the sum $A+B$ of two linear m-accretive operators A and B in a Hilbert space to be m-accretive. This condition is expressed in terms of $\mathrm{Re}(Au, Bu)$ for u in D, where D is a certain linear manifold contained in $D(A+B)$.

1. **Introduction.** A linear operator A (with domain $D(A)$ and range $R(A)$) in a (complex) Banach space X is said to be accretive[2] if

(A) $\qquad \|(A+\xi)u\| \geqq \xi\,\|u\| \quad$ for every $u \in D(A)$ and $\xi > 0$.

It is known that $R(A+\xi)=X$ either for every $\xi>0$ or for no $\xi>0$; in the former case we say that A is m-accretive. Let X^* be the adjoint space of X. The pairing between $w \in X$ and $f \in X^*$ will be denoted by (w,f). Let F be the duality map from X to X^*: $F(w)=\{f \in X^*;\ (w,f)=\|w\|^2=\|f\|^2\}$ for every $w \in X$. Then (A) is equivalent to the following condition:

(A′) For every $u \in D(A)$ there is $f \in F(u)$ such that $\mathrm{Re}(Au,f) \geqq 0$;

see [**6**] (in which the term "monotonic" was used instead of "accretive"). Note that the inequality is not required to hold for every $f \in F(u)$. But if A is m-accretive and densely defined, then the inequality holds for every $f \in F(u)$ (cf. [**8**, Remark 1 to Theorem 3.1]).

Now let X be reflexive. Then linear m-accretive operators in X are necessarily densely defined (see [**12**, p. 218]). So if A and B are linear m-accretive operators in X, then $A+B$ defined on $D(A+B)=D(A)\cap D(B)$ is always accretive according to condition (A′).

The main purpose of this note is to give a sufficient condition for the sum of two linear m-accretive operators in a Hilbert space to be m-accretive (§3). In particular we shall improve a perturbation theorem of Glimm

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[2] In this case $-A$ is said to be dissipative.

and Jaffe [3]. §2 contains a remark for (reflexive) Banach space case. We finally consider the relatively bounded perturbation of linear m-accretive operators in a Hilbert space (§4).

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2. **Singular perturbation in a reflexive Banach space.** Let A and B be linear m-accretive operators in a Banach space X and A_n be the Yosida approximation of $A: A_n = A(1+n^{-1}A)^{-1} = n(1-(1+n^{-1}A)^{-1}), n=1,2,\cdots$; A_n is a bounded linear operator on X. Since A_n is also accretive (see [**6**, Lemma 2.3]), it is easily seen that A_n+B defined on $D(B)$ is m-accretive. Consequently, for every $v \in X$ there exists a unique $u_n \in D(B)$ such that

$$A_n u_n + Bu_n + u_n = v, \qquad n = 1, 2, \cdots. \tag{2.1}$$

LEMMA 2.1. *Let X be reflexive and A be m-accretive in X.*

(a) *If $\{u_n\}$ is a sequence in $D(A)$ such that $u_n \rightharpoonup u \in X$ (we denote by $\rightharpoonup$ weak convergence) and if the $\|Au_n\|$ are bounded, then $u \in D(A)$ and $Au_n \rightharpoonup Au$.*

(b) *If $\{w_n\}$ is a sequence in X such that $w_n \rightharpoonup u \in X$ and if the $\|A_n w_n\|$ are bounded, then $u \in D(A)$ and $A_n w_n \rightharpoonup Au$.*

PROOF. (a) Since X is reflexive and the $\|Au_n\|$ are bounded, there is a subsequence $\{u_m\}$ of $\{u_n\}$ such that $Au_m \rightharpoonup v \in X$. Since A is closed, its graph is weakly closed so that $u \in D(A)$ and $v = Au$.

Since we could have started with any subsequence of $\{u_n\}$ instead of $\{u_n\}$ itself, the result obtained shows that $Au_n \rightharpoonup Au$.

(b) Set $u_n = (1+n^{-1}A)^{-1}w_n \in D(A)$. Then $Au_n = A_n w_n$ and the $\|Au_n\|$ are bounded. Also $w_n - u_n = (1-(1+n^{-1}A)^{-1})w_n = n^{-1}A_n w_n \to 0$ (we denote by $\to$ strong convergence) so that $u_n \rightharpoonup u$. Thus the result of (a) is applicable: $u \in D(A)$ and $A_n w_n = Au_n \rightharpoonup Au$. Q.E.D.

PROPOSITION 2.2. *Let X be reflexive. Let A and B be linear m-accretive operators in X. If $\|A_n(A_n+B+1)^{-1}\|$ is bounded as n tends to infinity, then $A+B$ is m-accretive.*

PROOF. Since $A+B$ is accretive, it suffices to show that $R(A+B+1) = X$. To see this, let u_n be the solution of equation (2.1). Since $\|u_n\| \leqq \|v\|$, there is a subsequence $\{u_m\}$ of $\{u_n\}$ such that $u_m \rightharpoonup u \in X$. Noting that $\|A_m u_m\|$ and $\|Bu_m\|$ are bounded in m by assumption, we see from Lemma 2.1 that $u \in D(A) \cap D(B)$, $Bu_m \rightharpoonup Bu$ and $A_m u_m \rightharpoonup Au$. Thus we obtain $v = (A+B+1)u$ which implies $R(A+B+1) = X$. Q.E.D.

REMARK 2.3. When X^* is uniformly convex, Proposition 2.2 holds even if A and B are nonlinear (see [7, Lemma 1]).

3. **Singular perturbation in a Hilbert space.** The following lemma is a linear version of a recent result of Brezis, Crandall and Pazy (see [**1**, Theorem 2.1]):

LEMMA 3.1. *Let A and B be linear m-accretive operators in a Hilbert space H. Then $R(A+B+1)=H$ if and only if $\|A_n(A_n+B+1)^{-1}\|$ is bounded as n tends to infinity.*

Note that the 'if' part of Lemma 3.1 is a particular case of Proposition 2.2. We shall use Lemma 3.1 in the following form:

LEMMA 3.2. *Let A and B be as in Lemma 3.1. Then $R(A+B+1)=H$ if and only if there are nonnegative constants a and $b<1$ such that*

$$0 \leqq 2\,\mathrm{Re}(A_n u, Bu) + a\,\|u\|^2 + b(\|A_n u\|^2 + \|Bu\|^2), \qquad u \in D(B). \tag{3.1}$$

PROOF. It is easily seen that (3.1) is equivalent to

$$\|A_n u\|^2 + \|Bu\|^2 \leqq c(\|(A_n + B)u\|^2 + \|u\|^2), \qquad u \in D(B),\ c \geqq 1. \tag{3.2}$$

Obviously, (3.2) implies that $\|A_n(A_n+B+1)^{-1}\|$ is bounded in n. Thus it suffices by Lemma 3.1 to show that if $\|A_n(A_n+B+1)^{-1}\|$ is bounded in n, then (3.2) holds. But since $\|B(A_n+B+1)^{-1}\|$ is also bounded in n, there exists a constant $c_1>0$ such that

$$\|A_n(A_n + B + 1)^{-1}v\|^2 + \|B(A_n + B + 1)^{-1}v\|^2 \leqq c_1\,\|v\|^2, \qquad v \in H. \tag{3.3}$$

Now let $u \in D(B)$. Setting $v=(A_n+B+1)u$ in (3.3), we can obtain (3.2). Q.E.D.

To simplify the statement, let us introduce the following

DEFINITION 3.3. A linear manifold D contained in the domain of a closed linear operator A is called a core of A if the closure of the restriction of A to D is again A.

Then our result is given by

THEOREM 3.4. *Let A and B be linear m-accretive operators in H. Let D be a linear manifold contained in $D(A)$ such that D is mapped into itself by $(1+n^{-1}A)^{-1}$ for $n=1, 2, \cdots$. Assume that D is contained in $D(B)$ and there exist nonnegative constants a and $b<1$ such that for all $u \in D$,*

$$0 \leqq 2\,\mathrm{Re}(Au, Bu) + a\,\|u\|^2 + b\,\|Au\|^2. \tag{3.4}$$

If D is a core of B, then $A+B$ is also m-accretive.

PROOF. Since $A+B$ is accretive, it suffices to show that $R(A+B+1)=H$. To see this, set $D_0=(1+A)^{-1}D$. Then D_0 is contained in D and

$\mathrm{Re}(Au, BAu) \geqq 0$ for $u \in D_0$. So, we see from (3.4) that for all $u \in D_0$,

$$0 \leqq 2\,\mathrm{Re}(Au, B(1+n^{-1})A)u) + a\,\|u\|^2 + b\,\|Au\|^2, \qquad n = 1, 2, \cdots. \tag{3.5}$$

Now let $v \in D$. Then $(1+n^{-1}A)^{-1}v \in D_0$ (use the resolvent equation). Setting $u=(1+n^{-1}A)^{-1}v$ in (3.5), we have that for all $v \in D$,

$$0 \leqq 2\,\mathrm{Re}(A_n v, Bv) + a\,\|v\|^2 + b\,\|A_n v\|^2.$$

Since D is a core of B, it follows that for all $u \in D(B)$, $0 \leqq 2\,\mathrm{Re}(A_n u, Bu) + a\,\|u\|^2 + b\,\|A_n u\|^2$. Thus, we see by Lemma 3.2 that $R(A+B+1)=H$.

Q.E.D.

Now set $D(A^\infty)=\bigcap_n D(A^n)$. Then $D(A^\infty)$ can be taken as D in Theorem 3.4 and we have

COROLLARY 3.5. *Let A and B be selfadjoint operators in H satisfying the inequality* (3.4) *with $b<1$ and $u \in D(A^\infty)$. If $D(A^\infty)$ is a core of B, then $A+B$ is also selfadjoint.*

PROOF. The selfadjointness of A and B implies that $\pm iA$ and $\pm iB$ are m-accretive. Since $(\pm iAu, \pm iBu)=(Au, Bu)$ for $u \in D(A^\infty)$, it follows from Theorem 3.4 that $\pm i(A+B)$ are m-accretive. But, $A+B$ is symmetric (note that $D(A^\infty)$ is dense in H) and so $A+B$ is selfadjoint. Q.E.D.

REMARK 3.6. Let A be a positive selfadjoint operator. Let B be a selfadjoint operator such that $D(A^\infty)$ is a core of B and $BD(A^\infty)$ is again contained in $D(A)$. Assume that there are nonnegative constants ε, α and β such that for $u \in D(A^\infty)$, $0 \leqq \alpha(Au, u)+(Bu, u)+\beta\|u\|^2$ and

$$0 \leqq \varepsilon\,\|Au\|^2 + (Bu, u) + ([A^{1/2}, [A^{1/2}, B]]u, u) + \beta\,\|u\|^3, \qquad 2\alpha + \varepsilon < 1,$$

where $[T, S]=TS-ST$ and $A^{1/2}$ is the square root of A. Then, since $[A^{1/2}, [A^{1/2}, B]]=[A^{1/2}, [A^{1/2}, B+\beta]]$, we have

$$0 \leqq 2\,\mathrm{Re}((A+\tfrac{1}{2})u, (B+\beta)u) + (2\alpha+\varepsilon)\,\|(A+\tfrac{1}{2})u\|^2, \qquad u \in D(A^\infty);$$

note that $\|Au\|^2 \leqq \|Au\|^2+(Au, u)+\|u\|^2/4=\|(A+\frac{1}{2})u\|^2$. Consequently, it follows from Corollary 3.5 that $A+B$ is selfadjoint. This fact is proved by Glimm and Jaffe under additional assumptions (see [**3**, Theorem 8]).

4. **Regular perturbation in a Hilbert space.** For the regular (relatively bounded) perturbation of linear m-accretive operators in a general Banach space see e.g. [**2**], [**4**], [**5**] and [**10**]. The following theorem is a variant of the result noted in [**9**]:

THEOREM 4.1. *Let A be a linear m-accretive operator in a Hilbert space H. Let B be a linear accretive operator in H with $D(B) \supset D(A)$. Assume that*

there are nonnegative constants a and $b \leqq 1$ such that

$$(4.1) \qquad 0 \leqq \operatorname{Re}(Au, Bu) + a\,\|u\|^2 + b\,\|Au\|^2, \qquad u \in D(A).$$

If $b<1$, then $A+B$ is also m-accretive. If $b=1$, then the closure of $A+B$ is m-accretive.

Note that (4.1) is similar to the inequality (3.4). First we prove

LEMMA 4.2. *If $b<1$ in Theorem 4.1, then $A+B$ is closed.*

PROOF. Since $D(A)$ is dense in H (as noted in §1), so is $D(B)$ and therefore B is closable (see [**5**, Theorem V-3.4]). Consequently, there exists a constant $c_1>0$ such that

$$(4.2) \qquad \|Bu\| \leqq c_1(\|u\| + \|Au\|), \qquad u \in D(A)$$

(see [**5**, Remark IV-1.5]). Also, we obtain from (4.1)

$$\begin{aligned} 0 &\leqq \operatorname{Re}(Au, Bu) + b\,\|(A + c)u\|^2 \\ &\leqq \operatorname{Re}((A + c)u, (A + B + c)u) + (b - 1)\,\|(A + c)u\|^2, \quad c = (a/b)^{1/2}. \end{aligned}$$

Hence it follows that

$$(4.3) \qquad (1 - b)\,\|(A + c)u\| \leqq \|(A + B + c)u\|, \qquad u \in D(A).$$

We see from (4.2) and (4.3) that $A+B$ is closed. Q.E.D.

PROOF OF THEOREM 4.1. It suffices by Lemma 4.2 to show that if $b \leqq 1$, then the closure of $A+B$ is m-accretive. Since $A+B$ is densely defined, it suffices to show that the adjoint $(A+B)^*$ of $A+B$ is accretive (cf. [**12**, p. 251]).

It follows from (4.1) that $0 \leqq \operatorname{Re}(Au, (A+B)u) + a\|u\|^2$ for $u \in D(A)$. Hence we obtain

$$(4.4) \quad 0 \leqq \operatorname{Re}((1 + n^{-1}A)u, (A + B)u) + n^{-1}a\,\|u\|^2, \qquad n = 1, 2, \cdots.$$

Now let $v \in H$. Then $(1+n^{-1}A)^{-1}v \in D(A)$. Setting $u=(1+n^{-1}A)^{-1}v$ in (4.4), we have that for all $v \in H$,

$$(4.5) \quad 0 \leqq \operatorname{Re}(v, (A + B)(1 + n^{-1}A)^{-1}v) + n^{-1}a\|(1 + n^{-1}A)^{-1}v\|^2.$$

Let $w \in D((A+B)^*)$. Then it follows from (4.5) that

$$0 \leqq \operatorname{Re}((A + B)^*w, (1 + n^{-1}A)^{-1}w) + n^{-1}a\|w\|^2, \qquad n = 1, 2, \cdots.$$

Going to the limit $n \to \infty$, we see that $(A+B)^*$ is accretive. Q.E.D.

REMARK 4.3. Theorem 4.1 is essentially proved by Yoshikawa though the closedness of B is assumed in his statement (see [**11**, Theorem 3.10 and Remark 3.12]).

REMARK 4.4. Let A and B be as in Theorem 4.1. Suppose that there are nonnegative constants a_1 and b_1 such that $\|Bu\|^2 \leqq a_1\|u\|^2 + b_1\|Au\|^2$ for $u \in D(A)$. If $b_1 \leqq 1$, then (4.1) holds with $b \leqq 1$. In fact, setting $a = a_1/2$ and $b = (b_1+1)/2$, we have that for $u \in D(A)$,

$$\begin{aligned}\|Bu\|^2 &\leqq 2a\,\|u\|^2 + (2b-1)\|Au\|^2 \\ &\leqq \|(A+B)u\|^2 - \|Au\|^2 + 2a\|u\|^2 + 2b\|Au\|^2 \\ &= \|Bu\|^2 + 2(\mathrm{Re}(Au, Bu) + a\|u\|^2 + b\|Au\|^2).\end{aligned}$$

Thus, Corollary 1 to Theorem 1 and Theorem 2 of [9] are covered by Theorem 4.1.

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