SHORTER NOTES

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A NEW PROOF OF A REGULARITY THEOREM FOR ELLIPTIC SYSTEMS

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Abstract. We give a proof, which makes use of the Riesz-Thorin theorem, for a smoothness theorem for solutions of elliptic systems in divergence form with bounded measurable coefficients. The results imply an important theorem in two dimensions due to Morrey [3]. Meyers has used a similar technique to get these results for elliptic equations [4].

Let \( \Omega \) be a compact domain in \( \mathbb{R}^n \) with smooth boundary and consider the \( m \times m \) real elliptic system given by:

\[
(Lu)_k = \sum_{i,j,k,l} \frac{\partial}{\partial x_i} a_{ij}^{kl} \frac{\partial}{\partial x_j} u_l
\]

where \( a_{ij}^{kl} \) are bounded measurable functions on \( \Omega \) satisfying the inequality

\[
\sum_{i,k,l} a_{ij}^{kl} \pi_i^l \pi_j^k \geq \sum_{i,k} (\pi_k^i)^2.
\]

\( E^p = H_0^1, p(\Omega) \) is the usual complex Sobolev space of \( m \) complex functions with \( p \)-integrable derivatives which are 0 on the boundary of \( \Omega \), and \( \| \cdot \|_p \) denotes norms in \( E^p \) or norms of operators on \( E^p \).

Theorem. There exist \( q_0 > 2 \) depending on \( \Omega, n \) and the number \( K = \max |a_{ij}^{kl} - \delta_i^k \delta_j^l| \) such that \( L : H^{1,q}(\Omega) \to H^{-1,q}(\Omega) \) is invertible for \( q_0 (q_0 - 1) < q < q_0 \).

Proof. Let \( \Delta \) be the Laplace operator on \( \Omega \), \( \Delta^{-1} \) its inverse.

\[
\Delta^{-1} : H^{1-p}(\Omega) \to H_0^{1,p}(\Omega)
\]

has norm \( c(p) < \infty \) for \( 1 < p < \infty \) [1]. We define an analytic family of operators

\[
A(\lambda) = \Delta^{-1}(L + \lambda \Delta) = \Delta^{-1}L + \lambda I.
\]
A(λ) is a bounded linear operator on $E^p$ for $1 < p < \infty$. To prove the theorem it is sufficient to show that $A(0)$ is invertible on $E^q$ for $q$ in a neighborhood of 2.

$A(\lambda)$ is invertible in $E^2$ for $\text{Re } \lambda > -1$ since we have

$$\text{Re} \langle u, A(\lambda)u \rangle_{E^2} = \text{Re} \langle u, \Delta u \rangle_{E^2} + \langle u, Lu \rangle_{E^2} \geq (1 + \text{Re } \lambda) \langle \nabla u, \nabla u \rangle_{E^2},$$

which shows that

$$\| A(\lambda)^{-1} \|_2 \leq 1/(1 + \text{Re } \lambda).$$

We also have that $A(\lambda)$ is invertible on $E^p$ for $|\lambda + 1| > c(p)K$, since

$$\| [\lambda + 1 - A(\lambda)]u \|_p \leq c(p)K \| u \|_p,$$

which shows that

$$\| A(\lambda)^{-1} \|_p \leq 1/(|\lambda + 1| - c(p)K).$$

We may apply an extension of the Riesz-Thorin theorem [2], with $\lambda = \lambda_1 x + \lambda_2 (1 - x)$ (all real), $1/q = x/2 + (1 - x)/p$, $\lambda_2 + 1 > c(p)K$ and $\lambda_1 > -1$ to get that $A(\lambda)$ is invertible on $E^2$ and

$$\| A(\lambda)^{-1} \|_q \leq \left( \frac{1}{\lambda_2 + 1 - c(p)K} \right)^{1-x} \left( \frac{1}{1 + \lambda_1} \right)^x.$$

If we choose $\lambda_1, \lambda_2, \alpha$ and $p$ properly, $\lambda = 0$ and $q$ lies in an interval about 2.

Corollary 1. If $n=2$ and $Lu = f$ for $u \in H^2_0(\Omega)$ and $f \in L^2(\Omega)$, then $u$ is Hölder continuous in $\Omega$.

Proof. Use the Sobolev imbedding theorems. $f \in H^{-1,q}(\Omega)$ and $u \in H^2_0(\Omega) \subset C^0(\Omega)$ for $(1 - \alpha)/2 = 1/q < 1/2$.

Corollary 2. If $K$ is sufficiently small, Corollary 1 holds if $n \neq 2$ and $f \in C^0(\Omega)$.

References


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