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NILPOTENT ELEMENTS IN BANACH ALGEBRAS

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ABSTRACT. Let $\mathfrak{A}$ be an A^*-algebra such that any maximal abelian *-subalgebra is regular and such that any quasinilpotent element x in $\mathfrak{A}$ satisfies $x^N=0$, with $N<\infty$. Then any irreducible Hilbert space *-representation is at most N-dimensional. If $\mathfrak{A}$ is a C^*-algebra, $\mathfrak{A}$ possesses transcendental quasinilpotent elements if there exists a $\pi \in \hat{\mathfrak{A}}$ with $\dim \pi = \infty$.

This note is a continuation of [1]. We want to establish an intimate connection between the degree of noncommutativity of a Banach algebra $\mathfrak{A}$ and the existence of nilpotent elements in $\mathfrak{A}$. The proofs are straightforward extensions of the matrix situation and of methods used in [1].

Throughout all Banach algebras are complex Banach algebras with norm $|\ |$, continuous involution $*$ and an auxiliary norm [3] $|\ |_0$ which satisfies $|xx^*|_0=|x|_0^2$. Such Banach algebras are called A^*-algebras [3]. In A^*-algebras maximal abelian *-subalgebras are necessarily semisimple. We shall further assume that all Banach algebras have a unit, because the adjunction of a unit does not affect the proofs nor the results. A representation π always denotes a *-representation on a Hilbert space H_π. We say that $x \in \mathfrak{A}$ is n-nilpotent, with $n=\infty, 1, 2, \cdots$, if x is quasinilpotent and $x^n=0$ but $x^{n-1}\neq 0$.

We begin with a simple lemma, which is probably known.

LEMMA. *Let $\mathfrak{A}$ be an A^*-algebra such that any $x=x^* \in \mathfrak{A}$ has a finite spectrum with at most n points; then* $\dim \mathfrak{A} \leqq n^2$.

PROOF. Let $\mathfrak{B}$ be a maximal abelian (semisimple)*-subalgebra of $\mathfrak{A}$. Then the character space $\hat{\mathfrak{B}}$ of $\mathfrak{B}$ has at most n elements. Hence any $x \in \mathfrak{B}$ has the form $x=\sum_{i\leqq n} \lambda_i e_i$, where the e_i are the minimal idempotents of $\mathfrak{B}$. The e_i are selfadjoint and it is easy to see that $\dim e_i\mathfrak{A}e_j \leqq 1$.

THEOREM 1. (i) *Let $\mathfrak{A}$ be an A^*-algebra such that any maximal abelian *-subalgebra is regular and such that any quasinilpotent element x in $\mathfrak{A}$ satisfies $x^N=0$, then any irreducible representation π of $\mathfrak{A}$ is at most N dimensional.*

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(ii) *Let $\mathfrak{A}$ be an A^*-algebra such that any maximal abelian $*$-subalgebra of $\mathfrak{A}$ is regular and assume $\mathfrak{A}$ has an irreducible representation π of dimension $1<N<\infty$, then $\mathfrak{A}$ has an N-nilpotent element.*

(iii) *Let $\mathfrak{A}$ be an A^*-algebra such that any irreducible representation of $\mathfrak{A}$ is at most N-dimensional; then any quasinilpotent element x in $\mathfrak{A}$ satisfies $x^N=0$.*

PROOF. (i) Let π be an irreducible representation of $\mathfrak{A}$ and let $\mathfrak{B}$ be a maximal abelian $*$-subalgebra of $\mathfrak{A}$. By ρ we denote the restriction of π to $\mathfrak{B}$. Then the hull $h(J)$ [3] of $J=\ker\rho$ is a closed set in $\hat{\mathfrak{B}}$. Assume we can find $N+1$ distinct points $t_1,\ldots,t_{N+1}\in h(J)$. Then there exist disjoint open neighborhoods U_i of t_i in $\hat{\mathfrak{B}}$. Since $\mathfrak{B}$ is regular we can find elements $a_i\in\mathfrak{B}$ with $\hat{a}_i(t_i)=1$ and $\hat{a}_i(t)=0$ for all $t\notin U_i$ with $1\leqq i\leqq N+1$. Here $\hat{}$ denotes the Gelfand transform. Then $a_ia_j=a_ja_i=0$ for $i\neq j$ and $a_i\notin\ker\pi$. Let $c_1,\ldots,c_N$ be arbitrary in $\mathfrak{A}$, then the element $a=a_1c_1a_2+\ldots+a_Nc_Na_{N+1}$ satisfies $a^{N+1}=0$. By assumption $a^N=a_1c_1a_2^2\cdots a_N^2c_Na_{N+1}=0$. This holds for all possible choices of c_i. Since $\pi(a_{N+1})\neq0$ there exists a $\xi\in H_\pi$ with $\pi(a_{N+1})\xi=\eta\neq0$. Then by assumption

$$\pi(a_1c_1a_2^2c_2\cdots a_N^2)\pi(c_N)\eta=0.$$

Keeping the $c_1,\ldots,c_{N-1}$ fixed for a moment, this holds for all c_N. However since π is irreducible the set $\{\pi(c_N)\eta|c_N\in\mathfrak{A}\}$ is dense in H_π. Hence $\pi(a_1c_1\cdots a_N^2)=0$ and this holds for any $(N-1)$-tuple $(c_1,\cdots,c_{N-1})$. Continuation of this argument finally leads to $\pi(a_1)=0$, a contradiction. Hence any selfadjoint element in $\pi(\mathfrak{A})$ has at most N points in its spectrum. By the lemma we get $\dim\pi\leqq N$.

(ii) Let π be an irreducible representation of dimension N, and let $e_{i,j}$, with $1\leqq i,j\leqq N$, be a system of matrix units in $\pi(\mathfrak{A})$. As in (i) we can find elements $a_i\in\mathfrak{A}$ such that $a_ia_j=a_ja_i=0$ for $i\neq j$ and $\pi(a_i)=e_{i,i}$. Further let $c_i\in\pi^{-1}(e_{i,i+1})$ then $a=a_1c_1a_2+\cdots+a_{N-1}c_{N-1}a_N$ satisfies $a^N=0$ and

$$\pi(a^{N-1})=\pi(a_1c_1a_2^2\cdots a_{N-1}^2c_{N-1}a_N)=e_{1,1}e_{1,2}e_{2,2}^2\cdots e_{N,N}=e_{1,N}\neq0.$$

(iii) Assume $\mathfrak{A}$ has a quasinilpotent element x with $x^N\neq0$.

Then there exists an irreducible representation π with $\pi(x^N)\neq0$. Since $\dim\pi\leqq N$ this is impossible.

For $N=1$ the theorem may be stated in slightly weaker form.

COROLLARY. *Let $\mathfrak{A}$ be an A^*-algebra with normal generators $\{x_i\}$ such that the abelian $*$-subalgebras $\mathfrak{B}_i$ generated by x_i and 1 are regular. Then $\mathfrak{A}$ is commutative iff $\mathfrak{A}$ has no nilpotent elements.*

PROOF. Assume $\mathfrak{A}$ has no nilpotent elements and let π be an irreducible representation of $\mathfrak{A}$. Then arguing as above we see that $\mathrm{Sp}(\pi x_i)$ consists of

only one point. Hence $\pi(x_i)=\lambda_i\mathbf{1}$ and $\dim \pi=1$. Thus $\mathfrak{A}$ is commutative. The converse is trivial, because commutative A^*-algebras are semisimple.

The corollary applies in particular to l^1-algebras of discrete groups [**1**].

Part (ii) of Theorem 1 can be extended to $n=\infty$ for C^*-algebras.

THEOREM 2. *Let $\mathfrak{A}$ be a C^*-algebra such that there is a $\pi\in\hat{\mathfrak{A}}$ with* $\dim \pi=\infty$. *Then $\mathfrak{A}$ has an ∞-nilpotent element.*

PROOF. Again there is no loss of generality to assume that $\mathfrak{A}$ has a unit. We assume further that $\mathfrak{A}$ has no ∞-nilpotent elements and derive a contradiction. Let $\pi\in\hat{\mathfrak{A}}$ be an infinite dimensional representation and assume there is a $b=b^*\in\pi(\mathfrak{A})$ such that the spectrum of b contains an interval $[\alpha,\beta]$ with $\beta>\alpha$. Using the functional calculus of C^*-algebras we find an $a=a^*\in\pi(\mathfrak{A})$ with $\operatorname{Sp} a=[0,1]$. Let $c\in\mathfrak{A}$ be positive hermitean with $\pi(c)=a$ and let f_i be the function.

$$\begin{aligned}
f_i(t) &= 0, & t &\leqq \frac{1}{i+1},\\
&= \text{linear}, & \frac{1}{i+1} &\leqq t \leqq \frac{3i+1}{3i(i+1)},\\
&= \frac{1}{i^2}, & \frac{3i+1}{3i(i+1)} &\leqq t \leqq \frac{3i+2}{(i+1)i3},\\
&= \text{linear}, & \frac{3i+2}{3i(i+1)} &\leqq t \leqq \frac{1}{i},\\
&= 0, & t &\geqq \frac{1}{i}.
\end{aligned}$$

Then the elements $a_i=f_i(c)$ satisfy $a_ia_j=0$ for $i\neq j$ and $\pi(a_i)=f_i(a)\neq 0$. Now let $q=\sum_{i=1} a_ic_ia_{i+1}$ with $c_i\in\mathfrak{A}$ and $|c_i|\leqq 2$. Then

$$q^k=\sum_{i=1}^{\infty} a_ic_ia_{i+1}^2\cdots c_{i+k-1}a_{i+k}$$

and

$$|q^k|\leqq\sum_{i=1}^{\infty}\frac{2^k}{i^2(i+1)^4\cdots(i+k)^2}\leqq\sum_{i=1}^{\infty}\frac{1}{i(i+1)\cdots(i+k)}=\frac{1}{kk!}.$$

Hence q is quasinilpotent for all possible choices of c_i. We shall show now that we can find $c_i\in\mathfrak{A}$ such that $|c_i|\leqq 2$ and such that $\pi(q^n)\neq 0$ for all n.

By construction of the a_i there exist unit vectors $\xi_i\in H_\pi$ with $\pi(a_i)\xi_i=(1/i^2)\xi_i$. Since π is irreducible there exist $c_i\in\pi(\mathfrak{A})$ with $|c_i|\leqq 2$ and $\pi(c_i)\xi_{i+1}=\xi_i$ [**2**, 2.8.3]. Let $\{c_i\}$ be a fixed sequence of such operators and

assume that the corresponding q satisfies $q^k=0$. Then

$$0 = \pi(q^k)\xi_{k+1} = \pi(a_1 \cdots a_k^2)\pi(c_k)\pi(a_{k+1})\xi_{k+1} = \cdots$$
$$= \frac{1}{(k+1)^2}\frac{1}{k^4}\cdots\frac{1}{2^4}\xi_1 \neq 0 \quad \text{a contradiction.}$$

The case when every maximal abelian subalgebra $\mathfrak{B}$ of $\pi(\mathfrak{A})$ has a totally disconnected spectrum can be treated similarly, because $\dim \pi = \infty$.

This theorem is an extension of a result of Topping [**4**], who showed that any antiliminal C^*-algebra possesses ∞-nilpotent elements. Our methods however are different and more direct. Theorem 2 shows that any C^*-algebra without ∞-nilpotent elements is necessarily liminal. This result is not optimal. To see this, let $\mathfrak{A}$ be a weak direct sum of liminal C^*-algebras $\mathfrak{A}_i$, $\mathfrak{A}=\sum' \oplus \mathfrak{A}_i$, such that any $\pi \in \hat{\mathfrak{A}}_i$ satisfies $\dim \pi < \infty$, but such that there exists a $\pi_i \in \hat{\mathfrak{A}}_i$ with $\dim \pi_i = p_i$ and $p_i \to \infty$. Then by Theorem 1 (ii) there exists a p_i-nilpotent element $a_i \in \mathfrak{A}_i$ with $|a_i|=1$. Then $a=\sum \oplus a_i \cdot 1/i$ is an ∞-nilpotent element. This example leads immediately to our next theorem.

THEOREM 3. *Let $\mathfrak{A}$ be a separable C^*-algebra without ∞-nilpotent elements. Then there exists a positive integer N such that any quasinilpotent element $x \in \mathfrak{A}$ satisfies $x^N=0$.*

PROOF. (a) By Theorem 2 we may assume $\mathfrak{A}$ to be liminal. Suppose further that $\mathfrak{A}$ has irreducible representations of arbitrary high but finite dimensions. Determine now as in [**2**, 4.4.4] an open set $\mathfrak{O}_1' \subset \hat{\mathfrak{A}}$ with the properties a, b and c of [**2**, 4.4.4]. Then there exist some positive integer n_1 and a nontrivial open subset $\mathfrak{O}_1 \subset \mathfrak{O}_1' \cap {}_{n_1}\hat{\mathfrak{A}}$ [**2**, 3.6.3] or any nontrivial open subset of $\mathfrak{O}_1'$ contains representations of arbitrary high dimensions. Similarly determine an open subset $\mathfrak{O}_2'$ in $\hat{\mathfrak{A}} \setminus {}_{n_1}\hat{\mathfrak{A}}$. Continuing this process we either find a sequence of nonempty open sets $\mathfrak{O}_i \subset {}_{n_i}\hat{\mathfrak{A}} - {}_{n_{i-1}}\hat{\mathfrak{A}}$ with $n_i \to \infty$ or $\hat{\mathfrak{A}}$ contains a locally compact open set $\mathfrak{O}$ such that any nontrivial open subset of $\mathfrak{O}$ contains representations of arbitrary high dimensions. In the first case let $J \subset \mathfrak{A}$ be the ideal corresponding to $\mathfrak{O}=\bigcup_{i=1}^{\infty} \mathfrak{O}_i$, then J has the form $J=\sum' \oplus J_i$ with $\hat{J}_i = \mathfrak{O}_i$ [**2**, 1.9.12]. Thus by our result above J contains an ∞-nilpotent element. In the second case determine an infinite family $\{\mathfrak{O}_i\}$ of nonempty disjoint open subsets of $\mathfrak{O}$. Let J_i be the ideal corresponding to $\mathfrak{O}_i$. Then the proof of Theorem 2 shows that J_i contains an i-nilpotent element x_i with $|x_i| \leqq 2^{-i}$. Clearly $x=\sum \oplus x_i$ is ∞-nilpotent.

An amusing consequence of Theorem 3 is the following result: Let $\mathfrak{A}$ be a separable C^*-algebra; then $\hat{\mathfrak{A}} = {}_n\hat{\mathfrak{A}}$ iff for any singly generated subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ we have $\hat{\mathfrak{B}} = {}_n\hat{\mathfrak{B}}$.

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