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ON THE EQUIVALENCE OF TWISTED GROUP ALGEBRAS AND BANACH *-ALGEBRAIC BUNDLES

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ABSTRACT. The relationship between twisted group algebras and Banach *-algebraic bundles is investigated. Informally stated, the results are that bundles with Borel cross sections correspond to twisted group algebras, and "locally continuous" twisted group algebras correspond to bundles. In the separable case, these results combine to give a complete correspondence between the bundles and the "locally continuous" algebras.

Fell, in [5], developed the theory of Banach *-algebraic bundles over a locally compact group. He was able to treat group extensions, the covariance algebras of quantum mechanics, the transformation group C^*-algebras of Glimm (see [6]) and Effros and Hahn (see [4]), and other examples as special cases, and he extended the Mackey-Blattner theory of induced representations to this general setting. On the other hand, Horst Leptin introduced the concept of generalized L^1-algebra in [7] and this concept was further developed in [1] under the name twisted group algebra. All the above mentioned examples can be treated in the context of twisted group algebras, and this has led to speculation concerning the precise relationship between these algebras and Fell's bundles. This problem has been mentioned by Fell in [5] and Leptin in [8]. In this note we will establish, in a rather simple way, much of this relationship. We need to point out here that the theory of twisted group algebras developed in [1] is restricted to the case of separable object algebra and second countable group. These restrictions are largely for convenience and are not used essentially until late in that paper. It is easy to see that the basic definitions and results are valid without the countability restrictions, and in fact Leptin develops completely analogous objects without such restrictions. Thus we have not hesitated to state our results in a nonseparable context, and we refer the reader to [7] for the techniques of generalization.

We will give a brief description of those parts of the above theories which we will use. We refer the reader to [1] and [5] for any definitions,

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proofs, or details which we do not give. Let A be a Banach *-algebra with a bounded, two-sided approximate identity, and let G be a locally compact group with left Haar measure m and modular function Δ. Let $U(A)$ and $\mathscr{A}(A)$ denote respectively the group of unitary double centralizers of norm one on A (with the strict topology), and the group of isometric *-automorphisms of A (with the topology of pointwise convergence). An element σ of $\mathscr{A}(A)$ lifts to $U(A)$ and we will denote the resulting automorphism of the group $U(A)$ by $\bar{\sigma}$.

DEFINITION 1. A twisting pair $\tau=(T, \alpha)$ for (G, A) consists of Borel mappings $T: G \to \mathscr{A}(A)$ and $\alpha: G \times G \to U(A)$ such that

(1) $[\overline{T(x)}\alpha(y, z)]\alpha(x, yz)=\alpha(x, y)\alpha(xy, z)\equiv\beta(x, y, z)$,

(2) $[T(x)T(y)a]\alpha(x, y)=\alpha(x, y)[T(xy)a]$,

(3) $T(e)a=a$, $\alpha(x, e)=\alpha(e, y)=I$,

where $x, y, z \in G$, $a \in A$, e and I are identities in G and $U(A)$ respectively.

If we replace elements $a \in A$ in the above equations by elements of $U(A)$ and replace each $T(x)$ by $\overline{T(x)}$, then the resulting pair $(\bar{T}, \alpha)$ is a factor system (in classical terminology) and corresponds to an algebraic extension

$$E(\bar{T}, \alpha): U(A) \overset{i}{\rightarrowtail} H \overset{\pi_0}{\twoheadrightarrow} G$$

of G by $U(A)$ ($\rightarrowtail$ denotes injection and $\twoheadrightarrow$ denotes surjection).

H is constructed by letting $H=U(A)\times G$ with multiplication defined as follows:

$$(u_1, g_1)\cdot(u_2, g_2) = (u_1(\bar{T}(g_1)u_2)\alpha(g_1, g_2), g_1g_2)$$

(see [**2**, §2]). Here $i(u)=(u, e)$ and $\pi_0(u, g)=g$.

DEFINITION 2. We say that (T, α) (resp. $(\bar{T}, \alpha)$) is smooth at the identity if:

(i) α is continuous at (e, e),

(ii) $(x, a)\to T(x)a$ (resp. $(x, u)\to\bar{T}(x)u$) is continuous at each point (e, a) of $G\times A$ (resp. (e, u) of $G\times U(A)$),

(iii) $\beta(x^{-1}, y, x)$ is continuous at $y=e$ for all x in G.

This definition is due to Calabi [**2**] for the pair $(\bar{T}, \alpha)$.

If $\tau=(T, \alpha)$ is a twisting pair for (G, A), then the space $L^1(G, A)$ of Bochner integrable, A-valued functions on G becomes a Banach *-algebra (the twisted group algebra $L^1(G, A; \tau)$ corresponding to G, A, and τ) if multiplication and involution are given respectively by:

$$(f\circ g)(x) = \int_G f(y)[T(y)g(y^{-1}x)]\alpha(y, y^{-1}x)\,dm(y)$$

$$f^*(x) = \alpha(x, x^{-1})^*[T(x)f(x^{-1})^*]\,\Delta(x)^{-1}$$

where f, g are in $L^1(G, A)$; x, y in G.

We refer the reader to [5] for the definitions of Banach *-algebraic bundle (abbreviated B-bundle from here on) and homogeneous B-bundle (abbreviated HB-bundle). Let $\mathscr{B}=(B, \pi, \cdot, *, G)$ be a fixed HB-bundle with approximate identity (π is the continuous open mapping of the bundle space B onto the locally compact group G; $\cdot$ and $*$ represent multiplication and involution respectively in B). Let $U(\mathscr{B})$ be the topological group (in general not locally compact) of unitary multipliers on $\mathscr{B}$, with the strong topology. There is a group extension

$$E: U(A) \overset{i}{\rightarrowtail} U(\mathscr{B}) \overset{\pi_0}{\twoheadrightarrow} G$$

where $A=B_e$, the unit fiber subalgebra of B, i is the canonical injection, and π_0 is the restriction of π to $U(\mathscr{B})$. We will call the extension $E \equiv E(\mathscr{B})$ the characteristic extension of $\mathscr{B}$. Fell has shown in [5, §9] that all HB-bundles can be constructed as follows:

Given A and G, let $E: U(A) \rightarrowtail^{i} H \twoheadrightarrow^{\pi_0} G$ be any topological extension of G by $U(A)$ and let $\phi: H \to \mathscr{A}(A)$ be such that:

(i) $h \to \phi(h)a$ is continuous on H for all a in A.

(ii) If $u \in U(A)$, $a \in A$, then $\phi(u)a = uau^{-1}$ (multiplication in the double centralizer algebra, $M(A)$, of A).

(iii) If $h \in H$ and $u \in U(A)$, then $\overline{\phi(h)}u = huh^{-1}$ (multiplication in H).

Define an equivalence relation $\sim$ in $A \times H$ by letting $(a_1, h_1) \sim (a_2, h_2) \Leftrightarrow$ there is a u in $U(A)$ such that $a_2 = a_1 u$ and $h_2 = u^{-1} h_1$. Let B be the quotient space $A \times H/\sim$ with the quotient topology, let $\pi: B \to G$ be the obvious continuous open projection, and let $\cdot$ and $*$ be defined in B by:

$$(a, h)^{\sim} \cdot (b, k)^{\sim} = (a \cdot \phi(h)b, hk)^{\sim} \quad \text{and}$$
$$((a, h)^{\sim})^* = (\phi(h^{-1})a^*, h^{-1})^{\sim} \quad \text{resp.}$$

Then $\mathscr{B}=(B, \pi, \cdot, *, G)$ is an HB-bundle and $E(\mathscr{B})=E$. All HB-bundles arise in this way.

DEFINITION 3. We will say that $\mathscr{B}$ is

(a) Borel, if there is a Borel cross section γ for

$$E(\mathscr{B})\ (\gamma: G \to U(\mathscr{B}); \pi_0 \gamma = \mathrm{Id}_G),$$

(b) smooth at the identity, if the above γ exists and is continuous at the identity in G.

If $\mathscr{B}=(B, \pi, \cdot, *, G)$ is a B-bundle, the cross sectional algebra $L^1(\mathscr{B})$ is the Banach *-algebra of all Borel sections f from G to B such that $\|f(x)\|$ is Haar integrable, and where multiplication and involution are given by:

$$(f \cdot g)(x) = \int_G f(y) \cdot g(y^{-1}x)\, dy$$

and

$$f^*(x) = f(x^{-1})^* \Delta(x)^{-1} \quad \text{respectively.}$$

THEOREM 1. *To each twisting pair $\tau=(T, \alpha)$ for (G, A), smooth at the identity, there corresponds an HB-bundle $\mathscr{B}_\tau \equiv (B_\tau, \pi, \cdot, *, G)$, smooth at the identity, in such a way that $L^1(G, A; \tau)$ and $L^1(\mathscr{B}_\tau)$ are isometrically *-isomorphic.*

PROOF. We form the extension $E(\bar{T}, \alpha): U(A) \rightarrowtail^i H \twoheadrightarrow^{\pi_0} G$ corresponding to the factor system $(\bar{T}, \alpha)$. We identify $U(A)$ with $i(U(A))$. Calabi has shown in [2] that (since $(\bar{T}, \alpha)$ is smooth at the identity) we may topologize H by taking product-neighborhoods as a base for the identity and translating these by elements of H to get the neighborhoods of other points. Then i is continuous and π_0 is continuous and open. Also $\gamma: G \to H$ given by $\gamma(g)=(I, g)$ is a Borel cross section continuous at the identity. We define $\phi: H \to \mathscr{A}(A)$ by letting $\phi(u, g)a = u(T(g)a)u^{-1}$. Then ϕ is a homomorphism. In fact:

$$\begin{aligned}
\phi(u_1, g_1) \cdot \phi(u_2, g_2)a \\
&= u_1(T(g_1)[u_2(T(g_2)a)u_2^{-1}])u_1^{-1} \\
&= u_1(T(g_1)u_2)(T(g_1)T(g_2)a)(T(g_1)u_2)^{-1}u_1^{-1} \\
&= u_1(T(g_1)u_2)\alpha(g_1, g_2)(T(g_1g_2)a)\alpha(g_1, g_2)^{-1}(T(g_1)u_2)^{-1}u_1^{-1} \\
&= \phi(u_1(T(g_1)u_2)\alpha(g_1, g_2), g_1g_2)a \\
&= \phi((u_1, g_1) \cdot (u_2g_2))a.
\end{aligned}$$

Since $g \to T(g)a$ is continuous at $g=e$, it follows that $(u, g) \to \phi(u, g)a$ is continuous at (I, e) and thus at all points of H. Furthermore:

(1) if $(u, e) \in U(A)$, $a \in A$ then $\phi(u, e)a = (uau^{-1}, e)$ and

(2) if $h \in H(h=(u_1, g_1))$, and $(u, e) \in U(A)$,

$$\bar{\phi}(u_1, g_1)(u, e) = (u_1, g_1)(u, e)(u_1, g_1)^{-1} = h(u, e)h^{-1}.$$

(2) is proved by a lengthy computation somewhat similar to the above proof that ϕ is a homomorphism. We omit the computation. We now have all the ingredients needed to construct an HB-bundle $\mathscr{B}_\tau = (B_\tau, \pi, \cdot, *, G)$. Computing explicitly in this case, we see that B_τ consists of all classes $(a, u, g)^\sim$ $(a \in A, u \in U(A), g \in G)$ where

$$(a_1, u_1, g_1) \sim (a_2, u_2, g_2) \Leftrightarrow g_1 = g_2 \text{ and } \exists\, u \in U(A)$$

such that $a_2 = a_1u$ and $u_2 = u^{-1}u_1$ and where

(i) $\pi[(a, u, g)^\sim] = g$,

(ii) $(a, u, g)^\sim \cdot (b, v, h)^\sim = (auT(g)b, \overline{(T(g)v)}\alpha(g, h), gh)^\sim$,

(iii) $((a, u, g)^\sim)^* = (T(g)^{-1}(u^{-1}a^*u), (T(g)u)\alpha(g^{-1}, g)^{-1}, g^{-1})^\sim$.

Let $F \in L^1(G, A; \tau)$ and define a section $\theta(F)$ on B_τ by:

$$\theta(F)(g) = (F(g), I, g)^{\sim}.$$

It is clear that θ is linear and that $\|\theta(F)\|_1 = \|F\|_1$, so θ is a linear isometry from $L^1(G, A; \tau)$ to $L^1(\mathscr{B})$. If $\check{F}$ is in $L^1(\mathscr{B})$ then $\check{F}(g) = (a, u, g)^{\sim} = (b, I, g)^{\sim}$ for a uniquely determined b. If we let $F(g) = b$, then clearly $F \in L^1(G, A; \tau)$ and $\theta(F) = \check{F}$, so θ is onto. Finally:

(i) If $F_1, F_2 \in L^1(G, A; \tau)$ then

$$\begin{aligned}
\theta(F_1) \cdot \theta(F_2)(x) &= \int_G \theta(F_1)(y) \cdot \theta(F_2)(y^{-1}x)\, dy \\
&= \int_G (F_1(y), I, y)^{\sim} \cdot (F_2(y^{-1}x), I, y^{-1}x)^{\sim}\, dy \\
&= \int_G (F_1(y)(T(y)F_2(y^{-1}x)), \alpha(y, y^{-1}x), x)^{\sim}\, dy \\
&= \int_G (F_1(y)(T(y)F_2(y^{-1}x))\alpha(y, y^{-1}x), I, x)^{\sim}\, dy \\
&= ((F_1^* F_2)(x), I, x)^{\sim} = \theta(F_1^* F_2).
\end{aligned}$$

(ii) If $F \in L^1(G, A; \tau)$ then

$$\begin{aligned}
\theta(F)^*(x) &= (\theta(F)(x^{-1}))^* \Delta(x)^{-1} \\
&= ((F(x^{-1}), I, x^{-1})^{\sim})^* \Delta(x)^{-1} \\
&= (T(x^{-1})^{-1}(F(x^{-1})^*), \alpha(x, x^{-1})^*, x)^{\sim} \Delta(x)^{-1} \\
&= (T(x^{-1})^{-1}(F(x^{-1})^*)\alpha(x, x^{-1})^* \Delta(x)^{-1}, I, x)^{\sim} \\
&= (\alpha(x, x^{-1})^{-1}(T(x)F(x^{-1}))^* \Delta(x)^{-1}, I, x)^{\sim} \\
&= (F^*(x), I, x)^{\sim} = \theta(F^*)(x).
\end{aligned}$$

Thus θ is an isometric *-isomorphism and Theorem 1 is proved.

THEOREM 2. *To each Borel HB-bundle $\mathscr{B}$ there is a twisted group algebra $L^1(G, A; \tau)$, which is isometrically isomorphic with $L^1(\mathscr{B})$.*

PROOF. Let $\mathscr{B} = (B, \pi, \cdot, *, G)$ be an HB-bundle with corresponding characteristic extension $E: U(A) \overset{i}{\rightarrowtail} U(\mathscr{B}) \overset{\pi_0}{\twoheadrightarrow} G$, where $A \equiv B_e$.

Let $\gamma: G \to U(\mathscr{B})$ be a Borel cross section for E. It is then well known that there is a factor system $(\bar{T}, \alpha)$ (where for $g_1, g_2 \in G$; $u \in U(A)$, $\bar{T}(g_1)u = \gamma(g_1)u\gamma(g_1)^{-1}$ and $\alpha(g_1, g_2) = \gamma(g_1)\gamma(g_2)\gamma(g_1g_2)^{-1}$) such that E is equivalent with $E(\bar{T}, \alpha)$ algebraically. Let $T(g)a = \gamma(g)a\gamma(g)^{-1}$ (which is defined since $\gamma(g) \in U(\mathscr{B})$). Then $\tau = (T, \alpha)$ is a twisting pair for (G, A) and $L^1(G, A; \tau)$ can be shown to be isometrically *-isomorphic with $L^1(\mathscr{B})$ exactly as this was done in the previous theorem.

To summarize the two preceding results informally, there is a correspondence between Borel HB-bundles and certain twisted group algebras which restricts to give a correspondence between all HB-bundles, smooth at the identity, and all twisted group algebras whose twisting pair is smooth at the identity.

Finally, we say that an HB-bundle $\mathscr{B}=(B, \pi, \cdot, *, G)$ is second countable if $U(\mathscr{B})$ is second countable (which is true if $A\equiv B_e$ is separable and G is second countable). In this case we know that there is a Borel cross section from G to $U(\mathscr{B})$, and thus every second countable HB-bundle corresponds to a twisted group algebra as above. We can, however, prove more:

THEOREM 3. *If $\mathscr{B}$ is second countable, then it is smooth at the identity.*

PROOF. Let $E: U(A) \rightarrowtail^{i} U(\mathscr{B}) \twoheadrightarrow^{\pi_0} G$ be the characteristic extension of $\mathscr{B}$. Nagao has shown in [9] that there is a cross section γ for E which is continuous at the identity, however γ is not necessarily Borel. We indicate a modification of Nago's proof which yields a Borel γ.

Let $(A_n)1 \leqq n < \infty$ be a strictly decreasing basis of open neighborhoods of the identity ε in $U(\mathscr{B})$, such that $A_1 = U(\mathscr{B})$. For each n let B_n be the saturation of A_n relative to coset decomposition mod $U(A)$ (i.e. $B_n = \pi_0^{-1}\pi_0(A_n)$). Then B_n is open for all n. Finally, let $C_n = A_n \cap (B_{n+1})'$ ($'$ denotes set complement). Now if S is an open set in C_n, there must be an open set O in $U(\mathscr{B})$ such that $O \cap C_n = S$, and since $C_n \subset A_n$ and A_n is open, we may assume that $O \subset A_n$. If x is in $\pi_0^{-1}\pi_0(O) \cap C_n$, then $\pi_0(x) = \pi_0(z)$ for some z in O but not in B_{n+1} ($\pi_0^{-1}\pi_0(B_{n+1}) = B_{n+1}$), and so z is in $O \cap (B_{n+1})' = S$. This means that $\pi_0^{-1}\pi_0(O) \cap C_n = \pi_0^{-1}\pi_0(S) \cap C_n$, and thus the saturation of S in C_n is open in C_n. The conditions of [3, Lemma 2] are seen to hold, since each C_n is a Polish space, and so we conclude that there is a Borel cross section γ_n from $\pi_0(C_n)$ to C_n for each n. The union of the disjoint Borel sets $\pi_0(C_n)$ is G, and thus there is a Borel section $\gamma: G \rightarrow U(\mathscr{B})$ such that $\gamma|(C_n) = \gamma_n$ for all n. It now follows as in [9] that γ is continuous at the identity in G.

COROLLARY. *In the second countable case there is a one-to-one correspondence between all HB-bundles and all twisted group algebras whose twisting pair is continuous at the identity. This correspondence takes a bundle $\mathscr{B}$ to an algebra $L^1(G, A; \tau)$ which is isometrically isomorphic with $L^1(\mathscr{B})$.*

REMARK. Whether or not, conversely, every separable twisted group algebra is isomorphic with the cross sectional algebra of an HB-bundle (i.e. whether or not every twisting pair is cohomologous to a twisting pair continuous at the identity) is not known. If, however, $U(A)$ is locally

compact (e.g. A is a group algebra or finite dimensional) then this is indeed the case, for one can use Weil's converse to Haar's theorem as Mackey did to get an extension of G by $U(A)$, and then proceed as in Theorem 3.

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