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A THEOREM ABOUT THE OSCILLATION OF SUMS OF INDEPENDENT RANDOM VARIABLES[1]

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ABSTRACT. Let $X_1, X_2, \cdots$ be i.i.d. random variables and let $S_n = X_1 + \cdots + X_n$. The relationship between the tth moment of X_1 and the convergence of the series $\sum_{n=1}^{\infty} z^n n^{t-1} P(S_n > 0)$ is investigated in this paper. The convergence of the series above when $|z| = 1$ but $z \neq 1$ is related to the oscillation of the sequence $\{P(S_n > 0)\}$ and to the oscillation of the sequence $\{S_n\}$ about zero.

Let $X, X_1, X_2, \cdots$ be i.i.d. random variables and define $S_n = X_1 + \cdots + X_n$ for $n = 1, 2, \cdots$. The main purpose of this paper is to prove the following theorem.

THEOREM. *If $t \geqq 1$ and if*

$$-\infty \leqq EX < 0 \quad \textit{and} \quad E(X^+)^t < \infty, \tag{1}$$

then for each $\delta > 0$ the series

$$\sum_{n=1}^{\infty} z^n n^{t-1} P\{S_n > 0\} \tag{2}$$

converges uniformly on the set

$$R_\delta = \{z : z \textit{ complex}, |z| \leqq 1, \textit{and } |1 - z| \geqq \delta\}. \tag{3}$$

Heyde [2] has shown that under the conditions of the theorem the series

$$\sum_{n=1}^{\infty} n^{t-2} P\{S_n > 0\} \tag{4}$$

converges. Hanson and Katz [1] have shown that even when there are

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no moment conditions imposed on X the series

$$\sum_{n=1}^{\infty}(-1)^n n^{-1}P\{S_n > 0\} \tag{5}$$

converges. (This might be considered to be the case $t=0$.) The fact that (5) converges can be interpreted as saying that the sequence of probabilities $P\{S_n>0\}$ does not oscillate too violently or that the sequence S_n does not oscillate too violently about zero. Note that if the sequence $P\{S_n>0\}$ converged monotonically to zero, then the convergence of (4) would imply the convergence of (2).

After proving the theorem some remarks will be made about the cases $t<1$.

Define

$$q_n = P\left\{\bigcap_{k=1}^{n}[S_k > 0]\right\} \quad \text{and} \quad p_n = P\{S_n > 0\} \quad \text{with } q_0 = 1. \tag{6}$$

We begin by giving four lemmas.

Lemma 1. *If* $t \geqq 1$ *and* (1) *holds, then, for all real* x,

$$\sum_{n=1}^{\infty} n^{t-2}P\{S_n > x\} < \infty. \tag{7}$$

Proof. The proof is, with minor changes, contained in the proofs of Lemmas 1 and 2 of Heyde [**2**].

Lemma 2. *If* $|z|<1$ *then*

$$\sum_{n=0}^{\infty} q_n z^n = \exp\left\{\sum_{n=1}^{\infty} p_n n^{-1} z^n\right\}. \tag{8}$$

Proof. See Corollary 2 in Spitzer [**6**] and the remarks immediately following Corollary 2.

Lemma 3. *If* $t \geqq 1$ *and* (1) *holds, then*

$$\sum_{n=1}^{\infty} n^{t-1} q_n < \infty. \tag{9}$$

Proof. If the right-hand side of (8) is written as a power series in z, then a comparison of the coefficients of z^n on both sides of equation (8) gives, for $n=1, 2, \cdots$,

$$q_n = \sum_{\nu=1}^{n} \frac{1}{\nu!} \sum\nolimits^{*} \prod_{i=1}^{\nu} k_i^{-1} p_{k_i} \tag{10}$$

where $\sum^*$ is taken over all ordered ν-tuples $(k_1, \cdots, k_\nu)$ of positive integers for which $k_1+\cdots+k_\nu=n$. If $k_1+\cdots+k_\nu=n$ and $k_i \geqq 1$ for $i=1, \cdots, \nu$, then $\prod_{i=1}^{\nu} k_i^{t-1} \geqq (n/\nu)^{t-1}$ so that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{t-1} q_n &\leqq \sum_{n=1}^{\infty} \sum_{\nu=1}^{n} (\nu^{t-1}/\nu!) \sum{}^* \prod_{i=1}^{\nu} k_i^{t-2} p_{k_i} \\ &= \sum_{\nu=1}^{\infty} (\nu^{t-1}/\nu!) \sum_{n=\nu}^{\infty} \sum{}^* \prod_{i=1}^{\nu} k_i^{t-2} p_{k_i} \\ &= \sum_{\nu=1}^{\infty} (\nu^{t-1}/\nu!) \left[\sum_{n=1}^{\infty} n^{t-2} p_n\right]^{\nu}.\end{aligned}$$

This last expression is finite from Lemma 1.

For convenience we define

$$A(z) = \sum_{n=1}^{\infty} p_n n^{-1} z^n. \tag{11}$$

LEMMA 4. *If* $\exp\{-A(z)\}$ *is written as a power series*

$$\exp\{-A(z)\} = \sum_{n=0}^{\infty} d_n z^n,$$

then $|d_n| \leqq q_n$ *for all* n.

PROOF. Note that $d_0=q_0=1$ and that, for $n \geqq 1$,

$$d_n = \sum_{\nu=1}^{n} \frac{(-1)^\nu}{\nu!} \sum{}^* \prod_{i=1}^{\nu} k_i^{-1} p_{k_i}.$$

Comparing this with (10) completes the proof.

PROOF OF THE THEOREM. Differentiating (8) and multiplying by $z[\exp\{-A(z)\}]$ gives, for $|z|<1$,

$$[\exp\{-A(z)\}] \sum_{n=1}^{\infty} n q_n z^n = \sum_{n=1}^{\infty} p_n z^n. \tag{12}$$

From Lemmas 3 and 4 the two series on the left-hand side of (12) each converge absolutely for $|z|<1$. Hence the product series converges absolutely for $|z|<1$ and, for $n=1, 2, \cdots$,

$$p_n = \sum_{k=0}^{n} (k q_k) d_{n-k}. \tag{13}$$

Fix $\delta>0$. The remainder of the proof consists of showing that (2) converges uniformly on R_δ. For $0<N_0\leqq N\leqq M$,

$$\sum_{n=N}^{M} z^n n^{t-1} p_n = \sum_{n=N}^{M} z^n n^{t-1} \sum_{k=0}^{n} d_k(n-k)q_{n-k}$$

$$= \sum_{k=0}^{M-N_0} d_k z^k \sum_{\nu=\max\{N-k,N_0\}}^{M-k} (\nu+k)^{t-1}\nu q_\nu z^\nu \tag{14}$$

$$+ \sum_{\nu=1}^{N_0-1} \nu q_\nu z^\nu \sum_{k=N-\nu}^{M-\nu} (\nu+k)^{t-1} d_k z^k. \tag{15}$$

Now $\{q_n\}$ is nonincreasing and, from Lemma 3, $q_n\to 0$ as $n\to\infty$. We define $\gamma_n=q_n-q_{n+1}$ so that $q_n=\sum_{k=n}^{\infty}\gamma_k$ and $\gamma_n\geqq 0$ for all n. Set $c_\nu=\nu(\nu+k)^{t-1}-(\nu-1)(\nu+k-1)^{t-1}$ so that $\nu(\nu+k)^{t-1}=\sum_{\alpha=1}^{\nu} c_\alpha$ for $\nu=1, 2, \cdots$. (We omit an index k in the definition of c_ν.) Then if $n\leqq m$,

$$\left|\sum_{\nu=n}^{m} \nu(\nu+k)^{t-1} q_\nu z^\nu\right| = \left|\sum_{\nu=n}^{m} \nu(\nu+k)^{t-1} z^\nu \sum_{j=\nu}^{\infty}\gamma_j\right|$$

$$= \left|\sum_{j=n}^{\infty}\gamma_j \sum_{\nu=n}^{\min\{m,j\}} \nu(\nu+k)^{t-1} z^\nu\right|$$

$$= \left|\sum_{j=n}^{\infty}\gamma_j \sum_{\nu=n}^{\min\{m,j\}} z^\nu \sum_{\alpha=1}^{\nu} c_\alpha\right|$$

$$\leqq \sum_{j=n}^{\infty}\gamma_j \sum_{\alpha=1}^{\min\{m,j\}} c_\alpha \left|\sum_{\nu=\max\{n,\alpha\}}^{\min\{m,j\}} z^\nu\right|. \tag{16}$$

For $z\in R_\delta$ we have $|\sum_{\nu=r}^{s} z^\nu|\leqq 2/\delta$ so that (16) is bounded by

$$(2/\delta)\sum_{j=n}^{\infty}\gamma_j\sum_{\alpha=1}^{j} c_\alpha = (2/\delta)\sum_{\nu=n}^{\infty}\gamma_\nu \nu(\nu+k)^{t-1}.$$

Thus if $z\in R_\delta$ and $0<N_0\leqq N\leqq M$ we see that the absolute value of (14) is bounded by

$$(2/\delta)\sum_{k=0}^{\infty}|d_k|\sum_{\nu=N_0}^{\infty}\nu(\nu+k)^{t-1}\gamma_\nu$$

$$\leqq (2/\delta)d_0\sum_{j=N_0}^{\infty} j^t\gamma_j \tag{17}$$

$$+ (2/\delta)2^{t-1}\sum_{k=1}^{\infty}k^{t-1}|d_k|\sum_{j=N_0}^{\infty} j^t\gamma_j. \tag{18}$$

Now $\sum_{k=1}^{\infty}k^{t-1}|d_k|<\infty$ from Lemmas 3 and 4. Thus if (as we will show next) $\sum_{j=1}^{\infty} j^t\gamma_j<\infty$, then both (17) and (18) will converge to zero as $N_0\to\infty$, and thus (14) will converge to zero uniformly in N and M

(satisfying $N_0<N<M$) as $N_0\to\infty$. We have

$$\sum_{j=1}^{\infty} j^t\gamma_j \leqq \sum_{j=1}^{\infty}\gamma_j \sum_{k=1}^{j} tk^{t-1} = \sum_{k=1}^{\infty} tk^{t-1}\sum_{j=k}^{\infty}\gamma_j$$
$$= \sum_{k=1}^{\infty} tk^{t-1}q_k < \infty.$$

For $|z|\leqq 1$ the absolute value of (15) is bounded by

$$\sum_{\nu=1}^{N_0-1} \nu q_\nu \sum_{k=N-\nu}^{\infty} (\nu+k)^{t-1}\,|d_k|.$$

If $2N_0\leqq N\leqq M$ then the above is bounded by

$$2^{t-1}\sum_{\nu=1}^{N_0-1} \nu q_\nu \sum_{k=N-N_0}^{\infty} k^{t-1}\,|d_k|.$$

By Lemmas 3 and 4, for each fixed N_0 the above expression tends to zero as $N\to\infty$.

If $\varepsilon>0$ we first choose N_0 large enough that (14) is less than $\varepsilon/2$ in absolute value for all N and M satisfying $N_0\leqq N\leqq M$. Then we choose $N_1\geqq 2N_0$ such that (15) is less than $\varepsilon/2$ in absolute value for all $N_1\leqq N\leqq M$.

Concerning a converse. An examination of its proof shows that the theorem remains valid if (1) is replaced by

$$\sum_{n=1}^{\infty} n^{t-2}P\{S_n>0\} < \infty. \tag{19}$$

Smith [5] gives for each $t\geqq 1$ a distribution such that (19) holds and such that $EX^+=EX^-=\infty$.

If it is assumed that $t>1$ and that $-\infty\leqq EX<0$, is the converse to the theorem true? (I.e., does the convergence of (2) uniformly on R_δ for every $\delta>0$ imply $E(X^+)^t<\infty$?) Note that once it is assumed that $EX<0$ then $E(X^+)<\infty$ so there is no converse to worry about when $t=1$.

The case $0<t<1$. If we assume $-\infty\leqq EX<0$ then $E(X^+)^1<\infty$. The theorem holds for $t=1$ and, from an inspection of the proof of the theorem, we find that the conclusion of the theorem is valid for $0<t<1$ also.

The series (19) is finite for all $t<1$. The following example shows that the conclusion of the theorem does not hold for all $t<1$.

Example. Let $X_1, X_2, \cdots$ be i.i.d. random variables with $P\{X_i=1\}=P\{X_i=-1\}=\frac{1}{2}$. Then

$$P\{S_n=0\} = 0, \quad n \text{ odd},$$
$$= (\tfrac{1}{2})^{2k}\binom{2k}{k}, \quad \text{if } n=2k \text{ for some } k,$$

and, for $n=1, 2, \cdots$, $P\{S_n>0\}=\frac{1}{2}[1-P\{S_n=0\}]$. Thus if $0<t<1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n n^{t-1}P\{S_n>0\}\\
&\qquad=\lim_{m\to\infty}\sum_{k=1}^{m}-\left[(2k-1)^{t-1}(\tfrac{1}{2})-(2k)^{t-1}\left(\frac{1}{2}-\frac{P\{S_{2k}=0\}}{2}\right)\right]\\
&\qquad\leqq\lim_{m\to\infty}-\sum_{k=1}^{m}\tfrac{1}{2}(2k)^{t-1}P\{S_{2k}=0\}\\
&\qquad=-\sum_{k=1}^{\infty}(2k)^{t-1}(\tfrac{1}{2})^{2k+1}\binom{2k}{k}.
\end{aligned}$$

Using Stirling's formula we find that there exist a positive constant C and an integer k_0 such that for $k\geqq k_0$

$$\binom{2k}{k}(\tfrac{1}{2})^{2k+1}(2k)^{t-1}\geqq Ck^{t-3/2}.$$

Hence the conclusion of the theorem does not hold if $t\geqq\frac{1}{2}$.

The following partial results hold when $0<t<1$.

PROPOSITION 1. *If* $0<t<1$ *and if*

$$\sum_{n=1}^{\infty}n^{t-1}\,|p_n-p_{n+1}|<\infty \tag{20}$$

then for every $\delta>0$ *the series* (2) *converges uniformly on* R_δ.

PROOF. According to Theorem 5.1.8 of Hille [3] the series (2) converges uniformly on R_δ if the partial sums $\sum_{n=1}^{m} z^n$ are uniformly bounded on R_δ, $\lim_{n\to\infty} n^{t-1}p_n=0$ and

$$\sum_{n=1}^{\infty}|n^{t-1}p_n-(n+1)^{t-1}p_{n+1}|<\infty.$$

It is easy to see that the first two conditions are satisfied. Now note that

$$\sum_{n=1}^{\infty}|n^{t-1}p_n-(n+1)^{t-1}p_{n+1}|$$

$$\leqq\sum_{n=1}^{\infty}n^{t-1}\,|p_n-p_{n+1}| \tag{21}$$

$$+\sum_{n=1}^{\infty}p_{n+1}\,|n^{t-1}-(n+1)^{t-1}|. \tag{22}$$

The series (21) converges by hypothesis. The series (22) converges since $(n+1)^\alpha - n^\alpha = O(n^{\alpha-1})$.

Now we define

$$l = P\left\{\bigcap_{k=1}^{\infty}[S_k > 0]\right\}, \qquad l' = P\left\{\bigcap_{k=1}^{\infty}[S_k \leqq 0]\right\}, \tag{23}$$

and

$$\begin{aligned} \Sigma_1 &= \sum_{n=1}^{\infty} P\{S_n \leqq 0,\, S_{n+1} > 0\}, \\ \Sigma_2 &= \sum_{n=1}^{\infty} P\{S_n > 0,\, S_{n+1} \leqq 0\}. \end{aligned} \tag{24}$$

LEMMA 5. *$\Sigma_1 < \infty$ if and only if $\Sigma_2 < \infty$.*

PROOF. Let I_A denote the indicator function of the set A. Define $I_n = I_{\{S_n > 0, S_{n+1} \leqq 0\}}$ and $J_n = I_{\{S_n \leqq 0, S_{n+1} > 0\}}$. Set $I = \sum_{n=1}^{\infty} I_n$ and $J = \sum_{n=1}^{\infty} J_n$. For any given point w note that $J(w)$ is the number of times the sequence $\{S_n(w)\}$ "crosses" zero in an upward direction, $I(w)$ is the number of downward crossings, $|J(w) - I(w)| \leqq 1$ if either $J(w)$ or $I(w)$ is finite, and $J(w) = \infty$ if and only if $I(w)$ is infinite. Note that $E(I) = \Sigma_2$, that $E(J) = \Sigma_1$, and that if either of these expected values is finite the other differs from it by at most one.

LEMMA 6. *If either $l > 0$ or $l' > 0$ (or both), then Σ_1 and Σ_2 are finite.*

PROOF. An immediate consequence of Lemma 2.4 of Rosenblatt [4].

PROPOSITION 2. *If $0 < t < 1$ and either $l > 0$ or $l' > 0$ (or both), then for every $\delta > 0$ the series (2) converges uniformly on R_δ.*

PROOF. Note that $\sum_{n=1}^{\infty} |p_n - p_{n+1}| \leqq \Sigma_1 + \Sigma_2$. Then apply Lemmas 5 and 6 and Proposition 1.

Note that when $t = 1$ this method, as well as that used to prove the theorem, may be used. In this case, $S_n \to -\infty$ and $l' > 0$.

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