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FORMAL 3-DEFORMATIONS OF 2-POLYHEDRA

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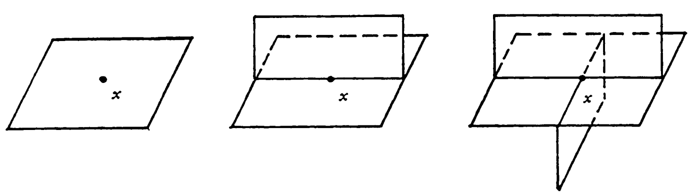
ABSTRACT. A formal deformation of one polyhedron to another is a finite sequence of expansions and collapses, beginning with one polyhedron and ending with the other. If a formal deformation exists between two 2-dimensional polyhedra, it is possible to choose a deformation through polyhedra of dimension at most four. It is desired to reduce this number to three. We give a partial result in that direction.

In [6] Whitehead introduced the concept of a formal deformation of a polyhedron. In terms of finite simplicial complexes, an elementary collapse $K \searrow L$ occurs if $L=K-\sigma^n-\tau^{n-1}$, where σ^n is a principal simplex of K and τ^{n-1} is a free face of σ^n. The inverse of this operation is called an elementary expansion. A formal deformation is a finite sequence of elementary collapses and expansions. Whitehead showed that if K_0 and K_1 are complexes of dimension n such that K_0 formally deforms to K_1, then K_0 $(n+2)$-deforms to K_1; that is, each complex in the sequence can be chosen to have dimension $\leq n+2$. This bound was reduced to $n+1$ by Wall [5] except for the case $n=2$. For 2-complexes the bound has not yet been reduced to three.

In this paper we consider the 2-polyhedra which have been called *closed fake surfaces* by Ikeda [3], and earlier called standard spines by Casler [2]. In such a (compact) polyhedron each point has a neighborhood of one of three types (Figure 1). The closed fake surfaces are more general than the standard spines, since the former need not embed in 3-manifolds.



Type I Type II Type III

FIGURE 1

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THEOREM 1. *Each compact 2-dimensional polyhedron formally 3-deforms to a closed fake surface.*

It was shown implicitly in [2] that Theorem 1 is true for any 2-polyhedron P in a 3-manifold M^3. That proof made extensive use of the surrounding manifold, and in fact the deformation of P to a closed fake surface was realizable in M^3.

PROOF OF THEOREM 1. Let P be a compact 2-polyhedron and T a triangulation of P.

Step 1. *Perform all available simplicial collapses.*

After each step we shall use the letters P and T again to denote the resulting polyhedron and some triangulation of it.

Write $T=X\cup Y$, where X is the subcomplex consisting of all 2-simplexes and their faces, and Y is the complementary principal 1-complex.

Step 2. *Deform P so that each* 1*-simplex of X has multiplicity two or three.* (That is, $\mathrm{lk}(\sigma^1, T)$ is a two or three point set.)

If $\sigma^1 \in X$, σ^1 has multiplicity at least two. If it is more than three, attach the ball pair (B^3, B^1) so that B^1 is identified with σ^1 and $B^3 \cap P$ is a relative regular neighborhood of σ^1 mod $\dot{\sigma}^1$ in T''. These new 3-cells intersect pairwise in either the empty set or a vertex of T. Collapse each 3-cell through any (maximal) free face. If T is any triangulation of the resulting polyhedron P, then T has the desired property.

Step 3. *Deform P so that the link of each vertex embeds in S^2, maintaining the property in Step* 2.

This occurs automatically in a 3-manifold.

Suppose $v \in T$ and $\mathrm{lk}(v, T)$ does not embed in S^2. Then $\mathrm{lk}(v, T)$ is a finite graph G and must contain one of the two classical nonplanar graphs [4]. Since each edge of T has multiplicity $\leqq 3$, so does each vertex of G. Thus G must contain the graph shown in Figure 2. Consider $\mathrm{lk}(v, T'')$, which is also homeomorphic to G. There is a simple closed curve J in $\mathrm{lk}(v, T'')$ which contains an arc A joining two vertices p, q of multiplicity 3 in $\mathrm{lk}(v, T'')$. Attach B^3 to P by identifying the boundary 2-cell D^2_- with the

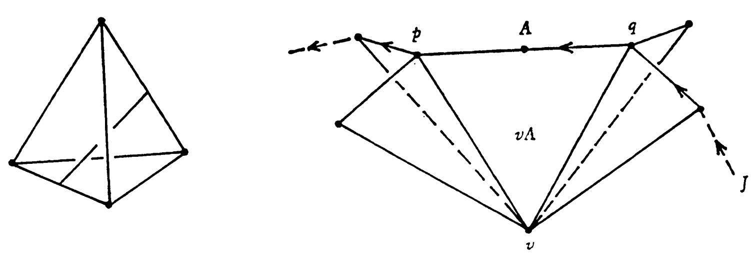

FIGURE 2

cone vJ, then collapse B^3 through the 2-cell vA. The following statements hold:

(i) A is a new edge of multiplicity 2.

(ii) $J-A$ consists of new edges of multiplicity 3.

(iii) vp and vq are new edges of multiplicity 2.

(iv) All other edges are unchanged, except for subdivision.

(v) The link of v in the new polyhedron is homeomorphic to $\mathrm{lk}(v, T'')-$ (interior of A), which has fewer edges than G.

(vi) For each vertex $w \neq v$ in T, the new link of w is homeomorphic to $\mathrm{lk}(w, T)$.

(vii) The links of all new vertices (they occur along J) embed in S^2.

Triangulate the new polyhedron in some manner. In a finite number of iterations of this process we deform P so that the link of v (in any triangulation) embeds in S^2.

The same process is then carried out at each vertex of the original triangulation T until all have links which embed in S^2. This process introduces many new vertices but each new vertex has an acceptable link which is unaltered by each succeeding iteration.

The new polyhedron P, with any new triangulation T, has the desired properties.

Step 4. *Thicken the vertices and the principal* 1*-simplexes of* T.

At each vertex $v \in T$, choose a p.l. embedding $f_v:\mathrm{lk}(v, T'')\to S^2$. Define $f_v(v)$ to be the center of B^3, and extend f_v conewise to a p.l. embedding $f_v:\mathrm{st}(v, T'')\to B^3$. Attach B^3 to P at each vertex $v \in T$ by the map f_v, and let N_v denote this 3-cell.

For each principal 1-simplex $\sigma \in T$, thicken $\mathrm{st}(\hat{\sigma}, T'')$ to a solid cylinder C_σ in such a way that, for each vertex v of σ, $C_\sigma \cap N_v$ is a 2-cell $E(v, \sigma)$ which misses the remaining $E(v, \sigma')$ and misses the rest of $\mathrm{lk}(v, T'')$ (in case $v \in X$).

Step 5. *Collapse* P *to a closed fake surface.*

The proof proceeds as in [2]. Each N_v can be collapsed to a house with two rooms H_v in such a way that the set of points of type II and III in H_v and the two boundary disks on N_v through which the collapse takes place do not intersect $\mathrm{lk}(v, T'')$ or any $E(v, \sigma)$. Then collapse each C_σ through either of its two ends $E(v, \sigma)$. The resulting polyhedron is a closed fake surface, and Theorem 1 is proved.

Not all closed fake surfaces embed in 3-manifolds. Ikeda's example of the wedge of circles a and b with two 2-cells attached by the words a, $ab^{-1}a^{-1}b^2$ is a contractible closed fake surface which does not embed in any 3-manifold. It contains a subpolyhedron Z homeomorphic to the quotient space of the product of a triod T with an interval, in which the ends $T\times\{0\}$ and $T\times\{1\}$ are identified by the permutation $(123)\to(132)$ of the three arcs in T. Any M^3 containing Z must contain a solid Klein bottle, but any 3-dimensional regular neighborhood of Ikeda's example would be contractible, hence orientable. Thus the presence of Z in a contractible fake surface is a geometric obstruction to such an embedding. Ikeda's example P does 3-deform to a point, however; in fact $P \times I$ is collapsible.

Conjecture. Each contractible closed fake surface 3-deforms to a spine of a 3-manifold.

A settlement of the conjecture in either direction has interesting consequences, summarized in the following:

THEOREM 2. *If the conjecture is true, then the* 3*-dimensional Poincaré conjecture implies each of the following statements*:

(1) *Each contractible* 2*-polyhedron* 3*-deforms to a point.*

(2) *The reduction of a presentation* $\{x_1, \cdots, x_n | r_1, \cdots, r_n\}$ *of the trivial group to the trivial presentation* $\{x_1, \cdots, x_n | x_1, \cdots, x_n\}$ *can be accomplished by the six operations described in* [**1**, paragraph 3].

(3) *The regular neighborhood of each contractible* 2*-polyhedron in* E^5 *is a ball.*

If the conjecture is false for some contractible closed fake surface P, *then* P *fails to* 3*-deform to a point. In particular, Zeeman's conjecture* (*that* P *contractible implies* $P \times I$ *collapsible*) *is false for* P.

PROOF OF THEOREM 2. Each contractible 2-polyhedron P 3-deforms to a contractible closed fake surface, by Theorem 1. If the conjecture is true, P further 3-deforms to the spine of a contractible 3-manifold M^3. If the 3-dimensional Poincaré conjecture is true, then $P \swarrow M^3 \searrow 0$, establishing (1). Statement (2) is equivalent to (1) and (3) follows from (2) (see [**1**]). The other statement of Theorem 2 is clear.

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