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DIRECT DECOMPOSITION OF TENSOR PRODUCTS INTO SUBTENSOR PRODUCTS

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ABSTRACT. A subtensor product of a family of modules is defined by using a subdirect product of the family of modules considered as sets. A tensor product of modules can be decomposed into a direct sum of subtensor products of the modules. Subtensor products of graded modules and graded algebras are also studied. As an application of these, a certain subtensor product of a family (not necessarily finite) of anticommutative algebras is shown to be a coproduct of this family in the category of unitary anticommutative algebras, and it can be imbedded as a direct summand into a tensor product of the family as modules.

1. **Subtensor product of modules.** Let $(M_\alpha)_{\alpha\in I}$ be a family of modules over a commutative ring R with unit, and U a subset of the cartesian product $\prod_{\alpha\in I} M_\alpha$ as sets. For an R-module N, a mapping $\varphi: U\to N$ will be called a *multilinear mapping of U into N* if for any $(x_\alpha)_{\alpha\in I}$, $(y_\alpha)_{\alpha\in I}$ and $(z_\alpha)_{\alpha\in I}$ in U such that $x_\beta=\lambda y_\beta+\mu z_\beta$, where λ and μ are in R, for one $\beta\in I$, and $x_\alpha=y_\alpha=z_\alpha$ for all $\alpha\in I$ with $\alpha\neq\beta$,

$$\varphi((x_\alpha)_{\alpha\in I}) = \lambda\varphi((y_\alpha)_{\alpha\in I}) + \mu\varphi((z_\alpha)_{\alpha\in I})$$

holds. A multilinear mapping of $\prod_{\alpha\in I} M_\alpha$ into N is a usual multilinear mapping; and a restriction mapping of this to U is an example of a multilinear mapping in our sense. By a similar construction to that of a tensor product of modules [**1**, Theorem 37, p. 87], the following theorem can be proved.

THEOREM 1. *For any $U\subseteq\prod_{\alpha\in I} M_\alpha$, there exist an R-module A and a multilinear mapping σ of U into A such that for any R-module N and for any multilinear mapping φ of U into N there exists a unique linear mapping f of A into N such that $f\circ\sigma=\varphi$.*

PROOF. Let F be a free R-module with U as its basis; and J be the submodule of F generated by the elements $(x_\alpha)_{\alpha\in I}-\lambda(y_\alpha)_{\alpha\in I}-\mu(z_\alpha)_{\alpha\in I}$, where

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$(x_\alpha)_{\alpha\in I}$, $(y_\alpha)_{\alpha\in I}$ and $(z_\alpha)_{\alpha\in I}$ are in U, λ, $\mu \in R$, such that $x_\beta=\lambda y_\beta+\mu z_\beta$ for one $\beta \in I$, and $x_\alpha=y_\alpha=z_\alpha$ for all $\alpha \in I$ with $\alpha\neq\beta$. Put $A=F/J$, and let σ be the restriction mapping of the natural homomorphism of F onto A to U. Q.E.D.

DEFINITIONS. Let $U\subseteq\prod_{\alpha\in I} M_\alpha$ be a subdirect product, i.e., for any $\beta \in I$, the mapping $U\to M_\beta$, $(x_\alpha)_{\alpha\in I}\mapsto x_\beta$ is onto. A pair (A, σ), where A and σ are as in Theorem 1, will be called a *subtensor product of* $(M_\alpha)_{\alpha\in I}$ *with respect to a subdirect product of* $(M_\alpha)_{\alpha\in I}$. A will be denoted by ${}_U\otimes_{\alpha\in I} M_\alpha$; and $\sigma((x_\alpha)_{\alpha\in I})$ by ${}_U\otimes\, x_\alpha$, or ${}_U\otimes_{\alpha\in I}\, x_\alpha$. f is called the *linearization of* φ.

REMARK 1. Let $(N_\alpha)_{\alpha\in I}$ be a family of R-modules such that M_α is a submodule of N_α for each $\alpha \in I$. If a subtensor product were defined for an arbitrary subset U of $\prod_{\alpha\in I} M_\alpha$, i.e., U is not necessarily a subdirect product, then ${}_U\otimes_{\alpha\in I} M_\alpha\cong{}_U\otimes_{\alpha\in I} N_\alpha$. This we do not want.

As usual, a subtensor product of $(M_\alpha)_{\alpha\in I}$ with respect to U is uniquely determined up to unique isomorphisms, and it is generated by the set of all elements ${}_U\otimes\, x_\alpha$ such that $(x_\alpha)_{\alpha\in I} \in U$.

2. **Direct decompositions into subtensor product.** Let U be a subset of $\prod_{\alpha\in I} M_\alpha$ satisfying the following condition (2.1).

(2.1) For any $(x_\alpha)_{\alpha\in I}$, $(y_\alpha)_{\alpha\in I}$, and $(z_\alpha)_{\alpha\in I}$ in $\prod_{\alpha\in I} M_\alpha$ such that $x_\beta=\lambda y_\beta+\mu z_\beta$, λ, $\mu \in R$, for one $\beta \in I$, and $x_\alpha=y_\alpha=z_\alpha$ for all $\alpha \in I$ with $\alpha\neq\beta$, if $(x_\alpha)_{\alpha\in I}$ is in U, both $(y_\alpha)_{\alpha\in I}$ and $(z_\alpha)_{\alpha\in I}$ are in U.

Then U is a subdirect product of $(M_\alpha)_{\alpha\in I}$. In fact, let $(x_\alpha)_{\alpha\in I} \in U$. For any $y \in M_\beta$, let $(y_\alpha)_{\alpha\in I}$ be an element of $\prod_{\alpha\in I} M_\alpha$ such that $y_\beta=y$ and $y_\alpha=x_\alpha$ for all $\alpha\neq\beta$; $(z_\alpha)_{\alpha\in I}$ such that $z_\beta=x_\beta-y$ and $z_\alpha=x_\alpha$ for all $\alpha\neq\beta$. Then $(y_\alpha)_{\alpha\in I} \in U$ by (2.1). It follows that the mapping $U\to M_\beta$, $(x_\alpha)_{\alpha\in I}\mapsto x_\beta$ is onto. Hence U is a subdirect product.

THEOREM 2. *Suppose that* $\prod_{\alpha\in I} M_\alpha=\bigcup_{j\in J} U_j$ *be a partition of* $\prod_{\alpha\in I} M_\alpha$ *into mutually disjoint subsets* U_j *satisfying* (2.1) *for each* $j \in J$. *Let* S_j *be the submodule of* $T=\otimes_{\alpha\in I} M_\alpha$ *generated by the elements* $\otimes\, x_\alpha$ *such that* $(x_\alpha)_{\alpha\in I} \in U_j$. *Then*

(1) $T=\sum_{j\in J} S_j$ *(direct)*,

(2) $S_j\cong{}_{U_j}\otimes_{\alpha\in I} M_\alpha$, *for each* $j \in J$.

More precisely, the linear mapping $q_j\colon {}_{U_j}\otimes_{\alpha\in I} M_\alpha\to T$, ${}_{U_j}\otimes\, x_\alpha\mapsto\otimes\, x_\alpha$, *is one-to-one and onto* S_j.

PROOF. Let $q\colon \oplus_{j\in J} ({}_{U_j}\otimes_{\alpha\in I} M_\alpha)\to T$ be the linear mapping defined by $q((y_j)_{j\in J})=\sum_{j\in J} q_j(y_j)$, where $y_j \in {}_{U_j}\otimes_{\alpha\in I} M_\alpha$. Also, for each $i \in J$, let $p_i\colon U_i\to \oplus_{j\in J} ({}_{U_j}\otimes_{\alpha\in I} M_\alpha)$ be the multilinear mapping defined by $p_i((x_\alpha)_{\alpha\in I})=(y_j)_{j\in J}$ such that $y_i={}_{U_i}\otimes\, x_\alpha$ and $y_j=0$ for all $j\neq i$. Let $p'\colon\prod_{\alpha\in I} M_\alpha\to \oplus_{j\in J} ({}_{U_j}\otimes_{\alpha\in I} M_\alpha)$ be the mapping extending each p_i, $i \in J$. p' is a multilinear mapping, since every U_j satisfies (2.1). If p is the linearization of p', then $q \circ p$ and $p \circ q$ are identity mappings. Hence q is an isomorphism. Also, it is clear that $q_j({}_{U_j}\otimes_{\alpha\in I} M_\alpha)=S_j$. (1) and (2) are immediate consequences of these. Q.E.D.

EXAMPLE 1. Define an equivalent relation on $\prod_{\alpha\in I} M_\alpha$ by $(w_\alpha)_{\alpha\in I}\sim(w'_\alpha)_{\alpha\in I}$ iff $w_\alpha=w'_\alpha$ for almost all $\alpha\in I$. Let $\Phi=\prod_{\alpha\in I} M_\alpha/\sim$, the quotient set. Then every $W\in\Phi$ satisfies (2.1). Hence, if we denote by S_W the submodule of T generated by the elements $\otimes x_\alpha$ such that $(x_\alpha)_{\alpha\in I}\in W$, for each $W\in\Phi$, it follows from Theorem 2 that

$$T=\sum_{W\in\Phi} S_W \quad \text{(direct)};$$

$$S_W\cong {}_W\otimes_{\alpha\in I} M_\alpha, \quad \text{for each } W\in\Phi.$$

LEMMA. *Let U be a subset of $\prod_{\alpha\in I} M_\alpha$; and let $\Phi'=\{W\in\Phi: W\cap U\neq\varnothing\}$, where Φ is as in Example* 1. *The following are equivalent.*

(1) *U satisfies* (2.1).

(2) $U=\bigcup_{W\in\Phi'} W$.

(3) *$U'=\prod_{\alpha\in I} M_\alpha-U$, the complementary set of U in $\prod_{\alpha\in I} M_\alpha$, satisfies* (2.1).

PROOF. Let $(w_\alpha)_{\alpha\in I}\in W\cap U$. Then for any $(x_\alpha)_{\alpha\in I}\in W$, $x_\alpha=w_\alpha$ for almost all $\alpha\in I$; say, $x_{\alpha_i}\neq w_{\alpha_i}$ for $i=1,2,\cdots,n$, and $x_\alpha=w_\alpha$ for all $\alpha\in I$ with $\alpha\neq\alpha_i$ for $i=1,2,\cdots,n$. Let $(y_\alpha)_{\alpha\in I}\in\prod_{\alpha\in I} M_\alpha$ such that $y_{\alpha_1}=x_{\alpha_1}$, $y_\alpha=w_\alpha$ for all $\alpha\in I$ with $\alpha\neq\alpha_1$; $(z_\alpha)_{\alpha\in I}$ such that $z_{\alpha_1}=w_{\alpha_1}-x_{\alpha_1}$, $z_\alpha=w_\alpha$ for all $\alpha\in I$ with $\alpha\neq\alpha_1$. Then $w_\alpha=y_\alpha+z_\alpha$ for one $\alpha=\alpha_1$, and $w_\alpha=y_\alpha=z_\alpha$ for all $\alpha\neq\alpha_1$. If U satisfies (2.1), $(y_\alpha)_{\alpha\in I}\in U$. Continue this process for $\alpha_2,\cdots,\alpha_n$. It follows from this that $(x_\alpha)_{\alpha\in I}\in U$. This implies that $W\subseteq U$ for all $W\in\Phi'$. Hence (1) implies (2). It is obvious that (2) implies (1). Similarly, we can prove that (2) if and only if (3).

THEOREM 3. *Let $I=\bigcup_{k\in K} J_k$ be a partition of I into mutually disjoint sets J_k; $\pi_k:\prod_{\alpha\in I} M_\alpha\to\prod_{\alpha\in J_k} M_\alpha$ a mapping defined by $\pi_k((x_\alpha)_{\alpha\in I})=(x_\alpha)_{\alpha\in J_k}$;* and *$\tau_k:\prod_{\alpha\in J_k} M_\alpha\to\otimes_{\alpha\in J_k} M_\alpha$ the universal multilinear mapping. Let U be a subset of $\prod_{\alpha\in I} M_\alpha$ satisfying* (2.1); *$U_k=\pi_k(U)$; $G_k=\tau_k(U_k)$; and $V=\{(y_k)_{k\in K}\in\prod_{k\in K}({}_{U_k}\otimes_{\alpha\in J_k} M_\alpha): y_k\in G_k$ for almost all $k\in K\}$. Then the R-homomorphism*

(2.2) *$\theta: {}_U\otimes_{\alpha\in I} M_\alpha\to{}_V\otimes_{k\in K}({}_{U_k}\otimes_{\alpha\in J_k} M_\alpha)$, such that*

$$ {}_U\otimes_{\alpha\in I} x_\alpha\mapsto{}_V\otimes_{k\in K}({}_{U_k}\otimes_{\alpha\in J_k} x_\alpha)$$

is an isomorphism.

PROOF. Let F_k be the free R-module with U_k as basis, for which ${}_{U_k}\otimes_{\alpha\in J_k} M_\alpha$ is a quotient in its construction (cf. Theorem 1), $\nu_k:F_k\to{}_{U_k}\otimes_{\alpha\in J_k} M_\alpha$ the natural homomorphism, and S_k the kernel of ν_k. We first consider the case where $U \in \Phi$ (Φ as in Example 1). Let

$$\bar{U} = \left\{(z_k)_{k\in K} \in \prod_{k\in K} F_k : z_k \in U_k \text{ for almost all } k \in K\right\}$$

and let

$$f : {}_{\bar{U}}\otimes_{k\in K} F_k \to {}_V\otimes_{k\in K} ({}_{U_k}\otimes_{\alpha\in J_k} M_\alpha)$$

be the linearization of $\bar{U} \to {}_V\otimes_{k\in K} ({}_{U_k}\otimes_{\alpha\in J_k} M_\alpha)$, $(z_k)_{k\in K} \mapsto {}_V\otimes_{k\in K} v_k(z_k)$. Let S be the submodule of ${}_{\bar{U}}\otimes_{k\in K} F_k$ generated by the elements ${}_{\bar{U}}\otimes_{k\in K} z_k$, where, for at least one $k \in K$, $z_k \in S_k$. Then $S \subseteq \operatorname{Ker} f$ clearly. We will show that $S = \operatorname{Ker} f$. Let ${}_{\bar{U}}\otimes z_k$, ${}_{\bar{U}}\otimes z'_k \in {}_{\bar{U}}\otimes_{k\in K} F_k$ such that $z_k - z'_k \in S_k$ for all $k \in K$. Since $U \in \Phi$ and z_k, $z'_k \in U_k$ for almost all $k \in K$, $z_k = z'_k$ for almost all $k \in K$. Let $z_{k_i} \neq z'_{k_i}$ for $i = 1, \cdots, n$. Then ${}_{\bar{U}}\otimes z_k - {}_{\bar{U}}\otimes z'_k = \sum_{i=1}^n w_i$, where $w_i = {}_{\bar{U}}\otimes_{k\in K} v_k$; $v_k = z_k = z'_k$ for $k \neq k_1, \cdots, k_n$; $v_{k_j} = z_{k_j}$ if $j < i$, $v_{k_i} = z_{k_i} - z'_{k_i}$, $v_{k_j} = z'_{k_j}$ if $j > i$. Since each $w_i \in S$, ${}_{\bar{U}}\otimes z_k - {}_{\bar{U}}\otimes z'_k \in S$. It follows from this that $V \to {}_{\bar{U}}\otimes_{k\in K} F_k/S$, $(y_k)_{k\in K} \mapsto {}_{\bar{U}}\otimes z_k + S$, where $z_k \in v_k^{-1}(y_k)$ for all $k \in K$, is a well-defined multilinear mapping. Let

$$g : {}_V\otimes_{k\in K} ({}_{U_k}\otimes_{\alpha\in J_k} M_\alpha) \to {}_{\bar{U}}\otimes_{k\in K} F_k/S$$

be the linearization of this, then $g \circ f = \nu$, where ν is the natural homomorphism of ${}_{\bar{U}}\otimes_{k\in K} F_k$ onto ${}_{\bar{U}}\otimes_{k\in K} F_k/S$. Hence $\operatorname{Ker} f = S$. Any element $z_k \in F_k$ can be uniquely expressed as $\sum_{i_k} r_{i_k}(x_{i_k,\alpha})_{\alpha\in J_k}$, where $r_{i_k} \in R$ and $(x_{i_k,\alpha})_{\alpha\in J_k} \in \prod_{\alpha\in J_k} M_\alpha$. Consider

$$\bar{U} \to {}_U\otimes_{\alpha\in I} M_\alpha, \qquad (z_k)_{k\in K} \mapsto \sum \left(\prod_k r_{i_k}\right) {}_{\bar{U}}\otimes_{\alpha\in I} x_{i_k,\alpha},$$

where the sum is over all $(i_k)_{k\in K}$. Then this mapping is properly defined, since $z_k \in U_k$ for almost all $k \in K$. Let h be the linearization of this. If $z_k \in S_k$,

$$z_k = \sum \pm((x_\alpha)_{\alpha\in J_k} - (y_\alpha)_{\alpha\in J_k} - (z_\alpha)_{\alpha\in J_k}),$$

where $x_\beta = y_\beta + z_\beta$ for one $\beta \in J_k$ and $x_\alpha = y_\alpha = z_\alpha$ for all $\alpha \in J_k$, $\alpha \neq \beta$. Hence if $(z_k)_{k\in K} \in S$, $h((z_k)_{k\in K}) = 0$. It follows that $h(S) = 0$. Thus, there exists an R-homomorphism $\eta; {}_V\otimes_{k\in K} ({}_{U_k}\otimes_{\alpha\in J_k} M_\alpha) \to \otimes_{\alpha\in I} M_\alpha$ such that $\theta \circ \eta$ and $\eta \circ \theta$ are identity mappings. We now consider the case where U is a subset of $\prod_{\alpha\in I} M_\alpha$ satisfying (2.1). Let $U = \bigcup_{W\in\Phi'} W$ (Φ' as in the Lemma). Then by Theorem 2, the Lemma, and the previous case,

$${}_U\otimes_{\alpha\in I} M_\alpha \cong \bigoplus_{W\in\Phi'} ({}_W\otimes_{\alpha\in I} M_\alpha) \cong \bigoplus_{W\in\Phi'} ({}_{V_W}\otimes_{k\in K} ({}_{W_k}\otimes_{\alpha\in J_k} M_\alpha)),$$

where W, W_k and V_W correspond to U, U_k and V in the previous case of $U \in \Phi$ respectively; and isomorphisms are canonical. ${}_{W_k}\otimes_{\alpha\in J_k} M_\alpha \subseteq {}_{U_k}\otimes_{\alpha\in J_k} M_\alpha$ and ${}_{V_W}\otimes_{k\in K} ({}_{W_k}\otimes_{\alpha\in J_k} M_\alpha) \subseteq {}_V\otimes_{k\in K} ({}_{U_k}\otimes_{\alpha\in J_k} M_\alpha)$ canonically, by Theorem 2, and the Lemma. V_W's are mutually disjoint, and hence

$$\bigoplus_{W\in\Phi'} ({}_{V_W}\otimes_{k\in K} ({}_{W_k}\otimes_{\alpha\in J_k} M_\alpha)) \subseteq {}_V\otimes_{k\in K} ({}_{U_k}\otimes_{\alpha\in J_k} M_\alpha)$$

canonically. Hence ${}_U\otimes_{\alpha\in I} M_\alpha$ can be canonically imbedded into ${}_V\otimes_{k\in K}({}_{U_k}\otimes_{\alpha\in J_k} M_\alpha)$. This canonical imbedding can be easily seen to be θ, hence θ is one-to-one. θ is clearly onto. Q.E.D.

EXAMPLE 2. Let $I=\bigcup_{k\in K} J_k$ be a partition of I into mutually disjoint sets J_k. If K is a finite set, it is well known [**1**, Theorem 39, p. 92] that

$$\bigotimes_{\alpha\in I} M_\alpha \cong \bigotimes_{k\in K} \left(\bigotimes_{\alpha\in J_k} M_\alpha\right) \tag{2.3}$$

by the canonical isomorphism. However, if K is infinite, the situation is slightly different. As a special case of Theorem 3, if we put $U=\prod_{\alpha\in I} M_\alpha$, it is easy to see that $\otimes_{\alpha\in I} M_\alpha$ is isomorphic to a direct summand of $\otimes_{k\in K} (\otimes_{\alpha\in J_k} M_\alpha)$.

Let $I=\bigcup_{k\in K} J_k=\bigcup_{k'\in K'} J'_{k'}$ be two partitions of I into mutually disjoint sets J_k and $J'_{k'}$ respectively. If K and K' are finite sets, it is clear from the isomorphism (2.3) that

$$\bigotimes_{k\in K} \left(\bigotimes_{\alpha\in J_k} M_\alpha\right) \cong \bigotimes_{k'\in K'} \left(\bigotimes_{\alpha\in J'_{k'}} M_\alpha\right) \tag{2.4}$$

canonically. However, if at least one of K and K' are infinite, (2.4) is no longer true. Let U be a subset of $\prod_{\alpha\in I} M_\alpha$ satisfying (2.1). Let V and U_k be as in Theorem 3, V' and $U'_{k'}$ be defined for the partition $\bigcup_{k'\in K'} J'_{k'}$ in the same way as V_k and U_k respectively.

COROLLARY 1. *Similar to* (2.4),

$${}_V\otimes_{k\in K} ({}_{U_k}\otimes_{\alpha\in J_k} M_\alpha) \cong {}_{V'}\otimes_{k'\in K'} ({}_{U'_{k'}}\otimes_{\alpha\in J'_{k'}} M_\alpha) \tag{2.5}$$

canonically.

3. **Subtensor product of graded modules.** In this section, we assume that, for each $\alpha\in I$, $M_\alpha=\sum_{\gamma\in\Gamma_\alpha} M_{\alpha.\gamma}$ (direct) is a graded R-module with Γ_α as its set of degrees; $M_{\alpha.\gamma}$ being the submodule of homogeneous elements of degree γ. For any $f\in\prod_{\alpha\in I}\Gamma_\alpha=P$, let T_f be the submodule of $T=\otimes_{\alpha\in I} M_\alpha$ generated by the elements $\otimes x_\alpha$ such that $x_\alpha\in M_{\alpha.f(\alpha)}$ for all $\alpha\in I$. If I is finite, it is well known [**1**, Theorem 40, p. 94] that

$$T=\sum_{f\in P} T_f \quad \text{(direct)}, \tag{3.1}$$

and that

$$T_f=\bigotimes_{\alpha\in I} M_{\alpha.f(\alpha)}, \quad \text{for each } f\in P.$$

Hence T is a graded module with P as a set of degrees. However, if I is infinite, (3.1) is not necessarily true.

Let T_H be the submodule of T generated by the elements $\otimes x_\alpha \in T$ such that x_α is homogeneous for all $\alpha \in I$ (notice that T_H is also generated by the elements $\otimes x_\alpha \in T$ such that x_α is homogeneous for *almost all* $\alpha \in I$); $T_{H'}$ generated by the elements $\otimes x_\alpha \in T$ such that x_α is not homogeneous for infinitely many $\alpha \in I$. Then by Theorem 2,

$$T = T_H + T_{H'} \quad \text{(direct)};$$
$$(3.2) \qquad T_H \cong {}_H\otimes_{\alpha\in I} M_\alpha \quad \text{and} \quad T_{H'} \cong {}_{H'}\otimes_{\alpha\in I} M_\alpha,$$

where $H=\{(x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} M_\alpha : x_\alpha$ is homogeneous for almost all $\alpha \in I\}$, and $H'=\{(x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} M_\alpha : x_\alpha$ is not homogeneous for infinitely many $\alpha \in I\}$.

THEOREM 4. *$T_H=\sum_{f\in P} T_f$ (direct), and $T_f \cong \otimes_{\alpha\in I} M_{\alpha,f(\alpha)}$, for each $f\in P$.*

PROOF. Let $p_{\alpha,f(\alpha)} : M_\alpha \to M_{\alpha,f(\alpha)}$ be the linear mapping such that the restriction of $p_{\alpha,f(\alpha)}$ to $M_{\alpha,\gamma}$, $\gamma \in \Gamma_\alpha$, is the identity mapping on $M_{\alpha,\gamma}$ if $\gamma=f(\alpha)$, zero for all $\gamma \neq f(\alpha)$. Suppose that p_f is the restriction mapping of $\otimes_{\alpha\in I} p_{\alpha,f(\alpha)}$ to T_H, and define $p: T_H \to \oplus_{f\in P} (\otimes_{\alpha\in I} M_{\alpha,f(\alpha)})$ by $p(x)=(p_f(x))_{f\in P}$ for $x \in T_H$. $p_f(x)=0$ for almost all $f \in P$, and hence p is properly defined. On the other hand, let $i_f : \otimes_{\alpha\in I} M_{\alpha,f(\alpha)} \to T_H$ be the linearization of the multilinear mapping $\prod_{\alpha\in I} M_{\alpha,f(\alpha)} \to T_H$, $(x_\alpha)_{\alpha\in I} \mapsto \otimes x_\alpha$. Let

$$i : \bigoplus_{f\in P} \left(\bigotimes_{\alpha\in I} M_{\alpha,f(\alpha)} \right) \to T_H$$

be a linear mapping defined by

$$i((y_f)_{f\in P}) = \sum_{f\in P} i_f(y_f), \qquad y_f \in \bigotimes_{\alpha\in I} M_{\alpha,f(\alpha)}.$$

$p \circ i$ and $i \circ p$ are identity mappings. Hence p is an isomorphism. (1) and (2) follow from this immediately. Q.E.D.

THEOREM 5. *Let U be a subset of H satisfying* (2.1), *where H is as in Theorem* 4. Then

$$(3.3) \qquad {}_U\otimes_{\alpha\in I} M_\alpha \cong \bigoplus_{f\in P} ({}_{U_f}\otimes_{\alpha\in I} M_{\alpha,f(\alpha)}),$$

where $U_f = U \cap \prod_{\alpha\in I} M_{\alpha,f(\alpha)}$. (This is a slightly generalized form of Theorem 4.)

PROOF. Essentially the same as the proof of Theorem 4.

4. **Generalized definition.** We would like to have a generalized definition of subtensor product so that ${}_{U_k}\otimes_{\alpha\in J_k} M_\alpha$ (in (2.2)), ${}_V\otimes_{k\in K} ({}_{U_k}\otimes_{\alpha\in J_k} M_\alpha)$

(in (2.2)) and ${}_{U_f}\bigotimes_{\alpha\in I} M_{\alpha,f(\alpha)}$ (in (3.3)) can be expressed as ${}_U\bigotimes_{\alpha\in J_k} M_\alpha$, ${}_U\bigotimes_{k\in K}({}_U\bigotimes_{\alpha\in J_k} M_\alpha)$ and ${}_U\bigotimes_{\alpha\in I} M_{\alpha,f(\alpha)}$ respectively. If we have one such, those properties which are true for a tensor product of a finite family also are true for a subtensor product of an infinite family. Namely, (2.2), (2.5) and (3.3) are

$$ {}_U\bigotimes_{\alpha\in I} M_\alpha \cong {}_U\bigotimes_{k\in K}({}_U\bigotimes_{\alpha\in J_k} M_\alpha), \tag{4.1}$$

$$ {}_U\bigotimes_{k\in K}({}_U\bigotimes_{\alpha\in J_k} M_\alpha) \cong {}_U\bigotimes_{k'\in K'}({}_U\bigotimes_{\alpha\in J'_{k'}} M_\alpha), \quad \text{and} \tag{4.2}$$

$$ {}_U\bigotimes_{\alpha\in I} M_\alpha \cong \bigoplus_{f\in P}({}_U\bigotimes_{\alpha\in I} M_{\alpha,f(\alpha)}), \tag{4.3}$$

respectively. The following definition serves the purpose.

DEFINITION. Let $(M_\alpha)_{\alpha\in I}$ be a family of R-modules, $J\subseteq I$ a subset, $J=\bigcup_{k\in K} J_k$ a partition of J into mutually disjoint subsets, N_k a submodule of $\bigotimes_{\alpha\in J_k} M_\alpha$ for each $k\in K$; $\pi_k:\prod_{\alpha\in I} M_\alpha\to\prod_{\alpha\in J_k} M_k$ a mapping defined by $\pi_k((x_\alpha)_{\alpha\in I})=(x_\alpha)_{\alpha\in J_k}$ for each $k\in K$, $\tau_k:\prod_{\alpha\in J_k} M_\alpha\to\bigotimes_{\alpha\in J_k} M_\alpha$ the universal multilinear mapping for each $k\in K$. Let U be a subset of $\prod_{\alpha\in I} M_\alpha$, $U_k=\pi_k(U)$, $G_k=\tau_k(U_k)$, and $V=\{(y_k)_{k\in K}\in\prod_{k\in K} N_k : y_k\in G_k\cap N_k\}$. ${}_V\bigotimes_{k\in K} N_k$ will be called a generalized *subtensor product of a family* $(N_k)_{k\in K}$ *with respect to a subdirect product* U *of* $(M_\alpha)_{\alpha\in I}$, and will be denoted by ${}_U\bigotimes_{k\in K} M_k$.

REMARK 2. We will demonstrate that ${}_{U_k}\bigotimes_{\alpha\in J_k} M_\alpha$ (in (2.2)) can be expressed as ${}_U\bigotimes_{\alpha\in J_k} M_\alpha$, in generalized definition. For any subset J of I, denote $\pi(U)$, where $\pi:\prod_{\alpha\in I} M_\alpha\to\prod_{\alpha\in J} M_\alpha$, $(x_\alpha)_{\alpha\in I}\mapsto(x_\alpha)_{\alpha\in J}$, by U_J. Consider the special case where $J=K$, $J_\alpha=\{\alpha\}$ for $\alpha\in K$, $N_\alpha=M_\alpha$ for $\alpha\in K$, then $V=\{(y_\alpha)_{\alpha\in J}\in\prod_{\alpha\in J} M_\alpha : y_\alpha\in G_\alpha\}=U_J$. Hence

$$ {}_{U_J}\bigotimes_{\alpha\in J} M_\alpha = {}_V\bigotimes_{k\in K} N_k = {}_U\bigotimes_{k\in K} N_k. \tag{4.4}$$

J_k is a subset of I. It follows from (4.4) that ${}_{U_k}\bigotimes_{\alpha\in J_k} M_\alpha={}_U\bigotimes_{\alpha\in J_k} M_\alpha$.

REMARK 3. The others follow similarly to Remark 2.

5. **Application: coproduct of anticommutative algebras.** Let $(A_\alpha)_{\alpha\in I}$ be a family of unitary graded algebras over a commutative ring R, all admitting the additive group Z of integers as their group of degrees. We shall denote by $A_{\alpha,k}$ the submodule of A_α of all homogeneous elements of degree k. In case I is finite, $T=\bigotimes_{\alpha\in I} A_\alpha$ is a graded R-algebra with Z as its group of degrees [**1**, pp. 154–156]. But if I is infinite, it is not possible to define a multiplication in T analogously to the finite case so that T is a graded algebra with Z as its group of degrees. However, we shall consider the submodule A of $T=\bigotimes_{\alpha\in I} A_\alpha$ generated by the set of all elements $\otimes a_\alpha$ in T such that $a_\alpha=1_\alpha$ for almost all $\alpha\in I$, where 1_α is the identity element of Z_α for each $\alpha\in I$.

DEFINITION. For a family $(A_\alpha)_{\alpha\in I}$ of unitary R-algebras, a subtensor product with respect to

$$W = \left\{(x_\alpha)_{\alpha\in I} \in \prod_{\alpha\in I} A_\alpha : x_\alpha = 1_\alpha \text{ for almost all } \alpha \in I\right\}$$

in the generalized definition will be called a *restricted tensor product* and will be denoted by $\otimes^w_{\alpha\in I} A_\alpha$.

In fact, $A \cong \otimes^w_{\alpha\in I} A_\alpha$, and $A = \sum_{f\in Q} S_f$ (direct) by Theorem 5, where $Q = \{f \in Z^I : f(\alpha) = 0 \text{ for almost all } \alpha \in I\}$, and $S_f, f \in Z^I$, is the submodule of A generated by the elements $\otimes\, a_\alpha \in A$ such that $a_\alpha \in A_{\alpha, f(\alpha)}$ for all $\alpha \in I$. $S_f = 0$ for $f \notin Q$, and $S_f \cong \otimes^w_{\alpha\in I} A_{\alpha,(f)\alpha}$.

We now define a multiplication on A so that A becomes a graded algebra. Let $\mu_\alpha : A_\alpha \otimes A_\alpha \to A_\alpha$ be the linearization of the multiplication in A_α; and for $f, g \in Q$, let $\mu_{\alpha,f,g}$ be the restriction of μ_α to $A_{\alpha,f(\alpha)} \otimes A_{\alpha,g(\alpha)}$; this last module considered as a submodule of $A_\alpha \otimes A_\alpha$. There is a canonical isomorphism $\omega_{f,g}$ of $S_f \otimes S_g$ onto $\otimes^w_{\alpha\in I} (A_{\alpha,f(\alpha)} \otimes A_{\alpha,g(\alpha)})$ which maps the element $(\otimes^w_{\alpha\in I} a_\alpha) \otimes (\otimes^w_{\alpha\in I} b_\alpha)$ to $\otimes^w_{\alpha\in I} (a_\alpha \otimes b_\alpha)$ (cf. (4.2)), considering that S_f is canonically identified with $\otimes^w_{\alpha\in I} A_{\alpha,f(\alpha)}$, for each $f \in Q$. Let I be totally ordered by $\leqq$, and set $N(f, g) = \sum_{\alpha>\beta} f(\alpha)g(\beta)$ ($N(f, g)$ is properly defined, since $f(\alpha)$ and $g(\alpha)$ are zero for almost all $\alpha \in I$), $\varphi_{f,g} = (-1)^{N(f,g)}\omega_{f,g}$, and $\mu_{f,g} = (\otimes^w_{\alpha\in I} \mu_{\alpha,f,g}) \circ \varphi_{f,g}$, where $\otimes^w_{\alpha\in I} \mu_{\alpha,f,g}$ is the restriction of $\otimes_{\alpha\in I} \mu_{\alpha,f,g}$ to $\otimes^w_{\alpha\in I} (A_{\alpha,f(\alpha)} \otimes A_{\alpha,g(\alpha)})$. Then $\mu_{f,g}$ is a linear mapping of $S_f \otimes S_g$ into S_{f+g}. We may assume that the modules $S_f \otimes S_g$ are taken to be submodules of $A \otimes A$ and that $A \otimes A$ is their direct sum (cf. (4.3)). Let μ be the linear mapping of $A \otimes A \to A$ which extends all the mappings $\mu_{f,g}$. If a, a' are any elements of A, we set

$$aa' = \mu(a \otimes a');$$

this gives a multiplication in A, and we have

$$\left(\overset{w}{\underset{\alpha\in I}{\otimes}} a_\alpha\right)\left(\overset{w}{\underset{\alpha\in I}{\otimes}} b_\alpha\right) = (-1)^{N(f,g)} \overset{w}{\underset{\alpha\in I}{\otimes}} (a_\alpha b_\alpha).$$

It can be shown that A is an R-algebra, in the same way as [**1**, pp. 155–156]. For any integer p, let $A_p = \sum_{s(f)=p} S_f$, where $s(f) = \sum_{\alpha\in I} f(\alpha)$. Then $A = \sum_p A_p$ (direct) is a graded algebra with Z as its group of degrees. Similarly to the proof of [**1**, Theorem 14, p. 161], it can be shown that if each A_α is an anticommutative algebra, then $A = \otimes^w_{\alpha\in I} A_\alpha$ is also an anticommutative algebra.

THEOREM 6. *For a family $(A_\alpha)_{\alpha\in I}$ of anticommutative R-algebras, a pair $(\otimes^w_{\alpha\in I} A_\alpha, (\theta_\alpha)_{\alpha\in I})$, where $\theta_\beta : A_\beta \to \otimes^w_{\alpha\in I} A_\alpha$ is a linear mapping such that $\theta_\beta(a) = \otimes\, a_\alpha$ with $a_\beta = a$, $a_\alpha = 1_\alpha$ for all $\alpha \in I$, $\alpha \neq \beta$, is a coproduct of $(A_\alpha)_{\alpha\in I}$*

in the category of unitary anticommutative R-algebras and unitary homogeneous homomorphisms of degree zero.

PROOF. Similar to the proof of [**2**, Theorem 1].

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